

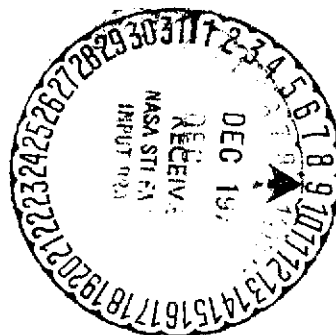
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APPLICATION OF WAVE MECHANICS THEORY TO FLUID DYNAMICS PROBLEMS

Boundary Layer on a Circular Cylinder Including Turbulence

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East Lansing, Michigan 48824



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Boundary Layer on a Circular Cylinder Including Turbulence

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INTRODUCTION

This report is concerned with the application of the elements of quantum (wave) mechanics to some special problems in the field of macroscopic fluid dynamics (mechanics), often referred to as classical fluid dynamics. In particular, considerations will be on flow of a viscous incompressible fluid around a circular cylinder. The presentation is divided into three sections: Section 1 constitutes a brief presentation of the flow of a nonviscous fluid around a circular cylinder. Attention is called to the fundamental concepts in any frictionless fluid flow such as velocity potential, stream function and so on. Section 2 presents a brief discussion of the restrictions imposed upon the stream function by the number of dimensions of space in which one usually operates (two in contrast to three), by the differences between stream function in two- and three-dimensional flows, and by the selection, as in the present case, of the two-dimensional space representation. The third section presents in detail the flow past three-dimensional bodies in a viscous fluid, particularly past a circular cylinder in the symmetrical case.

1. FLOW AROUND THREE-DIMENSIONAL BODIES--FRICTIONLESS FLUID FLOW

1.1. Characteristic Functions of the Flow

The part of fluid flow theory dealing with the characteristic functions of the non-viscid frictionless fluid flow with no heat conductivity is usually well understood by most scientists, physicists, engineers, and chemists and very little remains to be added. It appears desirable to call attention to a few of the most characteristic laws and functions which are valid and widely used in the field of frictionless fluid flows. Subsequently, these laws and functions are extended to and used in the field of viscous and heat-conductive fluid flows with possibly small modifications and adjustments, wherever they may become necessary. The most significant laws, rules and functions in the frictionless fluid flows are:

- a. conservation of mass (or the continuity equation);
- b. conservation of momentum;
- c. irrotationality;
- d. velocity potential;
- e. relation between the irrotationality and the existence of the potential function;
- f. conservation of energy.

The stream function requires a different definition for two-dimensional flows and a different one for three-dimensional flows. A streamline, however, is defined in the same manner for either two- or three-dimensional flow; viz., a continuous line through the fluid such that it has the direction of the velocity at every point throughout its length. The differential equation for a streamline in three-dimensional flow is:

$$dx/u = dy/v = dz/w; \quad (1.1.1)$$

whereas in two-dimensional flow the form is:

$$dx/u = dy/v. \quad (1.1.2)$$

In two-dimensional flow the concept of the stream function, $\psi = \psi(x, y)$, gives

$$u = -\partial\psi/\partial y; v = \partial\psi/\partial x; \partial u/\partial y = \partial v/\partial x; \quad (1.1.3)$$

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and finally for the irrotational flow case:

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = \nabla^2\psi = 0, \quad (1.1.4)$$

where the symbol ∇^2 denotes the Laplace operator (Laplacian).

2. SOME CHARACTERISTIC FUNCTIONS FOR TWO- AND THREE-DIMENSIONAL FLOWS

2.1. Stream Function in Two-Dimensional Flow

The stream function requires a different definition for two-dimensional flow from that for three-dimensional flow. One of the most useful definitions of the two-dimensional stream functions is that the partial derivative of the stream function with respect to any direction is the velocity component plus 90 degrees (counterclockwise) to the direction of flow (Streeter, p. 39). In addition it must satisfy the continuity equation. The first part of Equation (1.1.3), i.e., $u = \dots$, $v = \dots$, is true whether the flow has rotation or not. For irrotational flow, however, the second part of Equation (1.1.3) and Equation (1.1.4) are true. In particular, Equation (1.1.4) shows that the stream function may be constructed as the velocity potential for some other flow (Streeter, p. 39). Relations between stream function, ψ , and velocity potential, ϕ , are found by equating the expressions for the velocity components:

$$\partial\phi/\partial x = \partial\psi/\partial y; \quad \partial\phi/\partial y = -\partial\psi/\partial x. \quad (2.1.1)$$

2.2. Stream Function in Three-Dimensional Flow

Again, all of the details of three-dimensional flow will not be presented, only those which may be pertinent to the specific problem of the present report. The calculation of the stream function in three-dimensional flow requires much more effort and energy than that for the two-dimensional case. Of special interest and value is the Stokes' stream function defined only for those three-dimensional flow cases which have axial symmetry.

The above discussion demonstrates clearly the advantages of using the two-dimensional flow representation over three-dimensional flow representation, even in the cases where the actual flow in the physical space is geometrically expressed as a space phenomenon. The following representation of the flow phenomena in the viscous, incompressible

fluid in a three-dimensional space configuration will only use those analytical methods which give the mathematical description in terms of two independent variables.

3. FLOW PAST THREE-DIMENSIONAL BODIES IN VISCOUS INCOMPRESSIBLE FLUID

3.1. Flow Past a Circular Cylinder--Symmetrical Case

Flows past three-dimensional symmetrical bodies, in particular flow past a circular cylinder in a symmetrical configuration, will be the first case to be considered. The flow in the boundary layer past a circular cylinder in the symmetrical case is known from the literature, and the solution functions used here are taken from Schlichting, pp. 150-152. The free stream velocity U_∞ is parallel to x-axis, and one begins with the ideal velocity distribution in "potential" irrotational flow past a circular cylinder of radius R and free stream velocity U_∞ . In general, when one considers the flow past a cylinder in the symmetrical case, one may use the so-called Blasius series. According to Blasius, the potential flow is given by the series:

$$U(x) = u_1 x + u_3 x^3 + u_5 x^5 + u_7 x^7 + u_9 x^9 + u_{11} x^{11} + \dots; \quad (3.1.1)$$

where the coefficients u_2, u_3, \dots , depend only upon the shape of the body and are to be considered known. The pressure term in the boundary layer can be easily calculated for the stationary condition and becomes:

$$-\rho^{-1} dp/dx = U dU/dx; \quad \rho = \text{constant}; \quad (3.1.2)$$

$$\begin{aligned} U dU/dx = & u_1^2 x + 4 u_1 u_3 x^3 + x^5 (6 u_1 u_5 + 3 u_3^2) \\ & + x^7 (8 u_1 u_7 + 8 u_3 u_5) + x^9 (10 u_1 u_9 + 10 u_3 u_7 + 5 u_5^2) \\ & + x^{11} (12 u_1 u_{11} + 12 u_3 u_9 + 13 u_5 u_7) + \dots; \end{aligned} \quad (3.1.3)$$

where the continuity equation is integrated with the use of the stream function $\psi = \psi(x, y)$. A suitable assumption for the coefficients of the stream function and hence for the velocity components remains to be made. For the general case, Blasius proposed the form for $\psi(x, y)$, u and v with the standard boundary condition ($y = 0, u = v = 0, y = \infty$,

$$u = U_{\infty}$$

The case treated in this work refers to the boundary layer on a cylindrical body placed in a stream which is perpendicular to its axis (this axis will be denoted as the z-axis, perpendicular to the plane (x,y) in which the flow actually takes place). Thus the flow is considered to be two-dimensional in the (x,y) plane. In either case the velocity of the potential flow is assumed to have the form of a power series in x, where "x" denotes the distance from the stagnation point measured along the contour of the cylinder. In the Blasius series, the velocity profile in the boundary layer is also represented as a power-series in x, with the coefficients assumed to be functions of the coordinate y, measured at right angles to the wall; i. e., to the surface of the cylinder.

The form is:

$$U(x) = 2 U_{\infty} \sin \phi = 2 U_{\infty} \sin (x R^{-1}), \quad (3.1.4)$$

and by expanding in a power series it becomes:

$$U(x) = 2 U_{\infty} \left[(x R^{-1}) - \frac{1}{3!} (x R^{-1})^3 + \frac{1}{5!} (x R^{-1})^5 - \frac{1}{7!} (x R^{-1})^7 + \dots \right]; \quad (3.1.5)$$

the stream function with $\eta = y R^{-1} (2 U_{\infty} R \nu^{-1})^{1/2}$ is:

$$\begin{aligned} \psi = (\nu u_1^{-1})^{1/2} \{ & u_1 x f_1(\eta) + 4 u_3 x^3 f_3(\eta) + 6 u_5 x^5 f_5(\eta) \\ & + 8 u_7 x^7 f_7(\eta) + 10 u_9 x^9 f_9(\eta) + 12 u_{11} x^{11} f_{11}(\eta) + \dots \}; \end{aligned} \quad (3.1.6)$$

the functions u_n are:

$$u_1 = 2(U_{\infty} R^{-1}); \quad u_3 = -\frac{2}{3!} U_{\infty} R^{-3}; \quad u_5 = \frac{2}{5!} U_{\infty} R^{-5}; \dots; \quad (3.1.7)$$

and the velocity $u U_{\infty}^{-1}$ is given by:

$$\begin{aligned} u U_{\infty}^{-1} = 2 \{ & (x R^{-1}) f'_1 - \left(\frac{4}{3!}\right) (x R^{-1})^3 f'_3 + \left(\frac{6}{5!}\right) (x R^{-1})^5 f'_5 \\ & - \left(\frac{8}{7!}\right) (x R^{-1})^7 f'_7 + \dots \}. \end{aligned} \quad (3.1.8)$$

where the functions f'_n are called the functional coefficients.

In order to render the functional coefficients independent of the particular properties of the profile, i. e., of u_1, u_3, \dots , it is necessary to split them as follows (for $i \geq 5$):

$$f'_5 = g'_5 + (10/3) h'_5; \quad (3.1.9)$$

$$f'_7 = g'_7 + 7 h'_7 + (70/3) k'_7; \quad (3.1.10)$$

$$f'_9 = g'_9 + 12 h'_9 + (126/5) k'_9 + 84 j'_9 + 280 q'_9; \quad (3.1.11)$$

$$f'_{11} = g'_{11} + (155/3) h'_{11} + 66 k'_{11} + 220 j'_{11} + 462 q'_{11} \dots; \quad (3.1.12)$$

where all the functions shown are tabulated in Schlichting's work. The usefulness of the Blasius' method is restricted by the fact that it should not be used for slender bodies. Other attempts on the problem have been made by Goldstein, Hiemanz, Goertler and others. For a circular cylinder of diameter $d = 9.74$ cm in a stream of velocity $u_o = 19$ cm sec⁻¹, $Re = u_o d \nu^{-1} = 1.85 \times 10^4$, Hiemanz found that his experiment gave values for u which could be represented sufficiently accurately by three terms of the series for the potential flow. Due to the fact that the function $f_n(\eta)$ are tabulated in Schlichting's work, this research will use the Schlichting data throughout the report (see Schlichting, p. 152 to p. 154 after remodeling).

3.2. Forms Used in the Analysis

Stream function

$$\begin{aligned} \psi = (2 \nu U_\infty R)^{1/2} \{ & (R^{-1} x) f_1(\eta) - \frac{4}{3!} (R^{-1} x)^3 f_3(\eta) \\ & + \frac{6}{5!} (R^{-1} x)^5 f_5(\eta) - \frac{8}{7!} (R^{-1} x)^7 f_7(\eta) + \frac{10}{9!} (R^{-1} x)^9 f_9(\eta) \\ & - \frac{12}{11!} (R^{-1} x)^{11} f_{11}(\eta) + \dots \} . \end{aligned} \quad (3.2.1)$$

Velocity Components (Schlichting Data)

The velocity components are derived from the stream function:

$$u = (\partial \psi / \partial y) = (\partial \psi / \partial \eta) \cdot (2^{1/2} U_\infty^{1/2} R^{-1/2} \nu^{-1/2})$$

$$\begin{aligned}
&= 2 U_{\infty} \{ (xR^{-1}) f'_1(\eta) - \frac{4}{3!} (xR^{-1})^3 f'_3(\eta) + \frac{6}{5!} (xR^{-1})^5 f'_5(\eta) \\
&\quad - \frac{8}{7!} (xR^{-1})^7 f'_7(\eta) + \frac{10}{9!} (xR^{-1})^9 f'_9(\eta) \\
&\quad - \frac{12}{11!} (xR^{-1})^{11} f'_{11}(\eta) + \dots \} .
\end{aligned} \tag{3.2.2}$$

Similarly, the velocity component v is:

$$v = - \partial \psi / \partial x ; \tag{3.2.3}$$

$$\begin{aligned}
v = - (2 \nu U_{\infty} R)^{1/2} \{ R^{-1} f_1(\eta) - \frac{4}{2!} R^{-3} x^2 f_3(\eta) \\
+ \frac{6}{4!} R^{-5} x^4 f_5(\eta) - \frac{8}{6!} R^{-7} x^6 f_7(\eta) + \frac{10}{8!} R^{-9} x^8 f_9(\eta) \\
- \frac{12}{10!} R^{-11} x^{10} f_{11}(\eta) + \dots \} ; \quad \eta = (2 U_{\infty} R^{-1} \nu^{-1})^{1/2} y,
\end{aligned} \tag{3.2.4}$$

In Schlichting's work one can find another form for the function v , (p. 149, Equation(9.20a)) which, after inserting the corresponding values for the coefficients u_n , is in perfect agreement with Equation (3.2.4).

Final Forms of the Vorticity Component

$$\begin{aligned}
\partial v / \partial x = - (2 \nu U_{\infty} R)^{1/2} R^{-2} \{ - \frac{24}{3!} (xR^{-1}) f_3(\eta) \\
+ (\frac{120}{5!}) (xR^{-1})^3 f_5(\eta) - (\frac{336}{7!}) (xR^{-1})^5 f_7(\eta) \\
+ (\frac{720}{9!}) (xR^{-1})^7 f_9(\eta) - (\frac{1320}{11!}) (xR^{-1})^9 f_{11}(\eta) + \dots \}
\end{aligned} \tag{3.2.5}$$

$$\begin{aligned}
(\partial u / \partial y) = 2 U_{\infty} (2 R^{-1} U_{\infty} \nu^{-1})^{1/2} \{ (xR^{-1}) f''_1(\eta) \\
- (\frac{4}{3!}) (xR^{-1})^3 f''_3(\eta) + (\frac{6}{5!}) (xR^{-1})^5 f''_5(\eta) \\
- (\frac{8}{7!}) (xR^{-1})^7 f''_7(\eta) + (\frac{10}{9!}) (xR^{-1})^9 f''_9(\eta) \\
- (\frac{12}{11!}) (xR^{-1})^{11} f''_{11}(\eta) + \dots \} .
\end{aligned} \tag{3.2.6}$$

The function ω_z in the form

$$\omega_z = \frac{1}{2} [(\partial v / \partial x) - \partial u / \partial y], \approx \sec^{-1}, \quad (3.2.7)$$

will be superimposed upon each curve $u U_\infty^{-1}$ to illustrate the perturbed velocity distribution in the laminar boundary layer on a circular cylinder at the following set of angles:

$$\phi = 0^\circ; = 20^\circ; = 40^\circ; = 60^\circ; = 80^\circ; = 90^\circ; = 100^\circ; = 110^\circ. \quad (3.2.8)$$

where $\phi = (x R^{-1})$.

4. PERTURBATIONS

4.1. Stream Function

Thus, there have been derived the equations for the stream function, $\psi = \psi(x, \eta) = \psi(x, \eta(y))$, and the equations for the velocity components, (u, v) , in the specific flow in question, i.e., the symmetrical case of the flow past a circular cylinder (Blasius series). From the definition of the stream function one can derive the equation of the streamlines. Since, the stream function satisfies the equation of continuity, the curves, $\psi = \text{constant}$, represent streamlines (Owczarek, p. 63). The research begins with the stream function (see Equations (3.2.1) and (3.2.4) above):

$$\begin{aligned} \psi = (2 \nu U_\infty R)^{1/2} \{ & (R^{-1}x) f_1(\eta) - \frac{4}{3!} (R^{-1}x)^3 f_3(\eta) \\ & + \frac{6}{5!} (R^{-1}x)^5 f_5(\eta) - \frac{8}{7!} (R^{-1}x)^7 f_7(\eta) + \frac{10}{9!} (R^{-1}x)^9 f_9(\eta) \\ & - \frac{12}{11!} (R^{-1}x)^{11} f_{11}(\eta) + \dots \} = \text{constant}. \end{aligned} \quad (4.1.1)$$

Put in this form, the function ψ is a function of two variables, (x, η) , or (\bar{x}, η) , $\bar{x} = x R^{-1}$ (dimensionless), or (x, y) , since $\eta = \text{constant } y$, where the constant is a dimensional number. Thus, both the coordinates (\bar{x}, η) appearing in the representation of the function ψ , Equation (4.1.1) are dimensionless, and only the coefficient $(2 \nu U_\infty R)^{1/2}$ standing in front of $\{ \dots \}$ has the dimensions of $\text{cm}^2 \text{sec}^{-1}$ as it should be. The part of the stream function in the braces, $\{ \dots \}$, must be kept constant in order that the stream function be constant (condition for the streamlines). In other words, the condition for streamlines appears in the form:

$$\begin{aligned}
(R^{-1}x) f_1(\eta) - \frac{4}{3!} (R^{-1}x)^3 f_3(\eta) + \frac{6}{5!} (R^{-1}x)^5 f_5(\eta) \\
- \frac{8}{7!} (R^{-1}x)^7 f_7(\eta) + \dots = \text{constant};
\end{aligned} \tag{4.1.2}$$

and one may group the above terms in the following manner:

$$\begin{aligned}
[(R^{-1}x) f_1(\eta)] - [\frac{4}{3!} (R^{-1}x)^3 f_3(\eta)] + [\frac{6}{5!} (R^{-1}x)^5 f_5(\eta)] \\
- [\frac{8}{7!} (R^{-1}x)^7 f_7(\eta)] + [\frac{10}{9!} (R^{-1}x)^9 f_9(\eta)] - [\frac{12}{11!} (R^{-1}x)^{11} f_{11}(\eta)] \\
= \text{constant} = C.
\end{aligned} \tag{4.1.3}$$

Obviously, there are infinitely many possibilities for dividing the constant appearing on the right hand side of Equation (4.1.3) among the infinitely many terms of the series appearing on the left hand side of the same equation. For illustrative purposes, the following scheme is proposed: the constant C is divided into " m " equal parts, where " m " is equal to the number of terms on the left hand side of (4.1.3); each term is assumed to be equal to $C m^{-1}$; and $(R^{-1}x) = \bar{x}$:

$$\begin{aligned}
\bar{x} f_1(\eta) &= C m^{-1}; \\
- \frac{4}{3!} (\bar{x})^3 f_3(\eta) &= C m^{-1}; \\
\frac{6}{5!} (\bar{x})^5 f_5(\eta) &= C m^{-1}; \\
- \frac{8}{7!} (\bar{x})^7 f_7(\eta) &= C m^{-1}; \\
\frac{10}{9!} (\bar{x})^9 f_9(\eta) &= C m^{-1};
\end{aligned} \tag{4.1.4}$$

and so on

although there may be other alternatives, one of the conclusions of the system of equations, (4.1.4), is that, in the first form of (4.1.4), both the variables, (\bar{x}, η) , or any logical combination of these variables such as the product or the ratio are assumed to be constant. Again from the many possible combinations, which may and probably do occur in the actual, physical conditions, the following combination are selected for

purely illustrative purposes:

$$\bar{x} \eta = \text{constant} = C_A, \quad (4.1.5)$$

$$[(\bar{x})^n \eta] = \text{constant}; \eta = y R^{-1} (2 U_\infty R)^{-1/2}. \quad (4.1.6)$$

The first possibility, Equation (4.1.5), will be discussed in more detail. As can be seen from Equation (4.1.5) this is an equation of a hyperbola. Consequently, in the case under consideration, the simplest possible geometrical form of a streamline in the (x,y) space is a hyperbola, Equation (4.1.5).

4.2. Elementary Geometrical Characteristics of the Hyperbola

The geometrical characteristics of hyperbola are collected and presented in order to use them in the present work, thus eliminating the need for outside references. The standard form of a hyperbola is:

$$x^2 a^{-2} - y^2 b^{-2} = 1; \quad (4.2.1)$$

where the axis, x, intersects the hyperbola at the vertices A_1 and A_2 ; the segment $A_1 - A_2$ is the transverse axis; both branches of hyperbola are symmetrical with respect to $(A_1 A_2)$, one part being below and another above the axis; the y axis does not intersect the hyperbola since the center of the coordinates (x,y) is located in the center between A_1 and A_2 . From Equation (4.2.1), if $a = b$, the hyperbola is equilateral, and if the asymptotes are perpendicular to each other, the hyperbola is "rectangular." The equation of a rectangular hyperbola in which its asymptotes are referred to as coordinate axes is:

$$x y = \frac{1}{2} a^2 = A, \quad (4.2.2)$$

$$a^2 = b^2 = 2A,$$

and is located in the first and third quadrants. For such a hyperbola one has $x y = \frac{1}{2} a^2$ (first and third quadrants), $= -\frac{1}{2} a^2$ (second and fourth quadrants);

$$\text{semi-axes: } a = b = (2A)^{1/2}; \quad (4.2.3)$$

$$c^2 = a^2 + b^2 = 4A; \quad c = 2A^{1/2}; \quad (4.2.4)$$

$$\text{coordinates of foci: } (-2A^{1/2}, 0); (2A^{1/2}, 0); \quad (4.2.5)$$

$$\text{coordinates of vertices: } [(-2A)^{1/2}, 0]; [(2A)^{1/2}, 0]; \quad (4.2.6)$$

$$\text{eccentricity: } e = c a^{-1} = 2^{1/2}; \quad (4.2.7)$$

$$\text{distance of the center to directrix: } a e^{-1} = A^{1/2}; \quad (4.2.8)$$

$$\text{equation of directrices: } x = A^{1/2}, \text{ or } x = -A^{1/2}; \quad (4.2.9)$$

$$\text{equation of the asymptotes: } x - y = 0, x + y = 0; \text{ or } x = y, x = -y \quad (4.2.10)$$

Consequently, equation (4.1.5) is the equation of a rectangular hyperbola in which its asymptotes are referred to as dimensionless coordinate axes $(R^{-1}x, \eta)$ or (\bar{x}, η) . This coordinate system will be used as the basic coordinate system in further considerations, discussions and plotting of the diagrams.

4.3. Computer Plots

The plots, included in the present report refer to a solution of the steady-state boundary layer equation in the flow past a circular cylinder in the symmetrical case (Blasius series). In particular the plots refer to the horizontal velocity component in the case given by $u U_{\infty}^{-1}$ as quoted previously:

$$u U_{\infty}^{-1} = 2 \left\{ \bar{x} f'_1(\eta) - \frac{4}{3!} (\bar{x})^3 f'_3(\eta) + \dots \right\} \quad (4.3.1)$$

The plots of the function of $u U_{\infty}^{-1}$ as a function of two variables are given in the Schlichting book (p. 153) where the function $u U_{\infty}^{-1}$ is a function of $y R^{-1} (U_{\infty} R \nu^{-1})^{1/2} = (\eta 2^{-1/2})$ and of $\phi = x R^{-1}$, with ϕ being treated as a parameter in the range of $\phi = 0^\circ, 20^\circ, 40^\circ, 60^\circ, 80^\circ, 90^\circ, 100^\circ, 108.8^\circ$. Similar plots have been accomplished from this research which are done automatically in their entirety from the output plotter as part of the CDC 650 Computer System. These plots are functions of the independent variable η varying from $\eta = 0$ up to $\eta = 4.0$, and for the values of the parameter $\phi, \phi = x R^{-1}$:

$$\phi = 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ, 70^\circ, 80^\circ, 90^\circ, 100^\circ, 110^\circ.$$

Since this effort deals with the problem of the flow in a viscous

fluid (viscosity is defined as a transverse transfer of momentum) any possibility of obtaining a purely theoretical laminar flow around a cylinder is automatically excluded. One can only talk about a quasi-linear flow or a flow with small disturbances due to small viscosity phenomena. In order to make the disturbances due to viscosity visibly more observable, the concept of disturbance in the form of the vorticity function, ω_z , containing $(\partial v / \partial x)$, and $(\partial u / \partial y)$, has been introduced which is then superimposed upon the velocity function, $u U_\infty^{-1}$. This is done geometrically, by means of plotting the diagrams of $(u U_\infty^{-1})$ as the function of η in the two-dimensional Cartesian coordinate system, $(u U_\infty^{-1})$ and $\eta = y R^{-1} (2 U_\infty R \nu^{-1})^{1/2}$, with $\phi = \bar{x} = (x R^{-1})$ being a parameter, and R being a fixed constant. Moreover, the function ω_z is also plotted in the same diagram and in the same coordinate system as a function of η . Under the assumption that one is dealing with the linear Schroedinger equation as the basic equation, the principle of superposition can be applied and the values of the two functions $u U_\infty^{-1}$ and ω_z , and be added geometrically for each value of the independent variable, η . The final operation can then be performed for obtaining the sum of functions. This technique has been applied and the resulting plots are included as Appendix A of the report. It is not essential which coordinates in all of the systems discussed be chosen: (x, y) or (\bar{x}, y) , (\bar{x}, η) , (x, η) , since obviously the coordinates x and \bar{x} , y and η differ only by constants, one being dimensional and the other dimensionless, respectively. In conclusion, one may summarize the results obtained as follows:

- (a) the stream function of the flow around a circular cylinder, located symmetrically, is given by Equation (4.1.1);
- (b) the velocity component u is given by Equation (3.2.2);
- (c) the velocity component v is given by Equation (3.2.4);
- (d) the disturbance function (in form of the curl of velocity) is given by Equations (3.2.5), (3.2.6), and (3.2.7); this is (geometrically) superimposed upon the velocity component function $u U_\infty^{-1}$;
- (e) the condition for the streamlines is given by Equation (4.1.3);
- (f) the simplest possible geometrical form of a streamline in the (x, y) or (\bar{x}, η) space is the hyperbola $(\bar{x}\eta) = C_A$;

(g) consequently, the result of the case of flow around a circular cylinder is exactly the same as in the previous case of the laminar flow along an infinitely long plate. A finite but very small amount of vorticity introduced into the flow system causes the appearance of small disturbances. These disturbances originate at each finite element and spread in the fluid for short distances. In the case of the flat plate the streamlines were parabolas whereas in the present case they are hyperbolas as proven in Equation (4.1.5).

As in the case of a flat plate, one assumes a moving coordinate system (x, y) or rather $[(x R^{-1}), \eta]$ along each of the curves given as the sum of $[(u U_{\omega}^{-1}) + \omega_z]$, from Equation (3.2.2), plus Equation (3.2.7) with Equations (3.2.5) and (3.2.6), supplying a series of values of the parameter $x R^{-1} = \phi = 0^{\circ}, 20^{\circ}, \dots, 110^{\circ}$. Each chosen point for a certain value of η is assumed to be the vertex of the rectangular hyperbola. The moving coordinate system moves parallel to itself and to the fixed coordinate system, $[(x R^{-1}), \text{vertical axis}]$. The branches of the hyperbola are located in the first and third quadrants. From both coordinate axes, the horizontal axis $(R^{-1} x)$ is always horizontal and the other axis is always vertical. By means of the proper transformation (of coordinates if necessary) the branch of hyperbola located in the third quadrant is being shifted along the transverse axis $(A_1 - A_2)$ so that both vertexes A_1 and A_2 coincide at the vertex of the branch of the hyperbola located in the first quadrant. Consequently, the first quadrant branch of hyperbola is oriented from up-down and to the right; the third quadrant branch of hyperbola is oriented from down-up and to the left; both branches meet at their vertexes (A_1 at the same point as A_2) and both vertexes coincide with the point chosen on the curve $[(u U_{\omega}^{-1}) + \omega_z]$. The two branched hyperbolas in question are traced at a number of points along the curves $[(u U_{\omega}^{-1}) + \omega_z]$ and their points of intersections are located. The sections of streamlines (hyperbolas) between two found points of intersection furnish the zig-zag pattern of the path of a particle. This produces the resulting pattern of disturbances in the flow around a circular cylinder caused by the injection of the vorticity geometry into the regular pattern. These zig-zag patterns are traced in a few cases. The data used in the computer plots are the same referred to previously:

$$U_{\infty} = 200 \text{ km hour}^{-1} = 5555.55 \text{ cm sec}^{-1};$$

$$\nu = 0.149 \text{ cm}^2 \text{ sec}^{-1}.$$

4.4. The Plots

The first set of plots refers to the flow around a circular cylinder in the symmetrical case. The velocity distribution, $(u U_{\infty}^{-1})$, is taken as a function of $(x R^{-1})$ for various values of the coordinate η , considered to be a parameter. The value of η for 20 values is taken to be $(\eta = y R^{-1} (2U_{\infty}R\nu^{-1})^{1/2})$:

$$\eta = 0.2, 0.4, 0.6, \dots, \dots, 3.8 = 4.0; \quad (4.4.1)$$

In this approach no disturbances are superimposed upon the flow. The 20 plots included at the end of the present report are based on the values of the function of $(u U_{\infty}^{-1})$ as taken from Equation (3.1.8) or (4.3.1). For a certain number of points of these functions the streamlines are plotted as double-branched hyperbolas, and the points of intersections of these hyperbolas are found; the sections of streamlines between the points of intersection are interconnected thereby displaying the zig-zag pattern as the path of a particle. The results show that the zig-zag pattern, which indicates the existence of some disturbances in the boundary layer, appears even in the flow without the introduction of any outside disturbance such as vorticity. This can be expected in this kind of flow. It seems desirable to point out that one may question the accuracy and the precision of the graphical results and of the plots in the neighborhood of $x R^{-1}$ equal to 2.0. This may be partly justified since it must be kept in mind that: (a) the present calculations were done primarily for illustrative purposes, and for providing general conclusions giving insight into the particular points and peculiarities of the problem needing more emphasis, and (b) the point in question ($R^{-1} x = 2$) lies outside the region of the separation of the flow and consequently the characteristic properties of the flow at this particular point cannot be measured with precision.

The second set of plots is based on the curve $u U_{\infty}^{-1}$ plus a disturbance with resultant curve:

$$u U_{\infty}^{-1} + \omega_z = \text{resultant curve}; \quad (4.4.2)$$

on each curve ($u U_{\infty}^{-1}$) taken from the first set of plots there is superimposed the disturbance curve in the form of ω_z as the function of η . Each graph therefore clearly shows two curves: one describing the value of $u U_{\infty}^{-1}$ as the function of η and the second curve describing the vorticity function, ω_z , again as the function of η . The vertical coordinates on these two curves are geometrically added giving the resultant curve, Equation (4.4.2). Again at a certain number of points on this curve, the two-branched hyperbolas are plotted and the points of intersections of the streamlines (i. e. , of hyperbolas) are located. The zig-zag patterns of the paths of particles are traced thus giving the final results.

The third set of plots refers to the velocity v_d (dimensionless) = $v U_{\infty}^{-1}$, Equation (3.2.4), which is subject to disturbances in the form of the vorticity, ω_z , i. e. :

$$v_d = v U_{\infty}^{-1}; v \text{ from Equation (3.2.4); } v_d + \omega_z = \text{resultant function} \quad (4.4.3)$$

Again, at a certain number of points of the curve ($v_d + \omega_z$) two-branched hyperbolas are plotted, the points of intersections of the streamlines (i. e. , of hyperbolas) are located, and the zig-zag patterns of the paths of particles are traced, thus giving the final pattern.

All the plots described above and included in this report demonstrate the existence of the zig-zag patterns in the flow in question, regardless of whether the flow is laminar, irrotational, rotational, or there is or is not a vorticity function geometrically superimposed upon the velocity functions of u and v components. The existence of the zig-zag pattern is proof that the flow is not laminar, the particle paths are not parallel (nor quasi-parallel) lines. This is an obvious indication that there are disturbances in the particular flow in question due to the shape (circular) of the body around which the flow takes place. The laminar flow along a flat plate demonstrates the zig-zag pattern. There does not exist a laminar flow in a domain of a viscous, heat-conducting fluid flow (above the λ -transition point). This conclusion, which is in accordance with Heisenberg's statement, has been discussed in previous reports. The justification for the use of summation law, association between the wave mechanics and deterministic macroscopic fluid dynamics, and all the other aspects are also explained in previous reports.

5. FINAL REMARKS

A comparison of the geometry of the flow patterns of the laminar flow past a symmetrically located cylinder (Schlichting, pp. 146-155) and those in Technical Report No. 3 indicates a few important results:

- (1) The so-called "laminar" flow around a cylinder (symmetrical case or not) is not a laminar but is a flow with disturbances;
- (2) Disturbances are always present in the so-called "laminar" flow due to the transverse transfer of momentum (viscosity);
- (3) The zig-zag paths appear always in the so-called "laminar" flow, however small they may be since streamlines are always there;
- (4) The "streamlines" in the "laminar" flow are actually the mean value paths of the real zig-zag paths of the particles; due to the fact that zig-zag paths are small (small amplitudes) and due to the physiological aspects they are seen by the naked human eye as continuous lines (the streamlines);
- (5) The above traced zig-zag paths seem to be very "regular" whereas the oscillograms obtained from the oscillographs placed in a turbulent jet (see Technical Reports No. 1 and 2) demonstrate that often the realistic zig-zag paths are irregular; a certain regularity in zig-zag paths may or may not appear in the "periodic" sense. This demonstrates that "local" irregularities (jumps, sharp steep "mountains", sharp, steep "valleys") have their origins and are due to other reasons and phenomena not yet discussed (interference, inter-correlation);
- (6) When the zig-zag paths are very "even", regular and of small amplitudes, a zig-zag path may be seen by the naked human eye as one thick "streamline", as one thick ray; a light, a ray of light, which is also a wave;
- (7) The problem of interference and inter-correlation is discussed in Technical Report No. 1.

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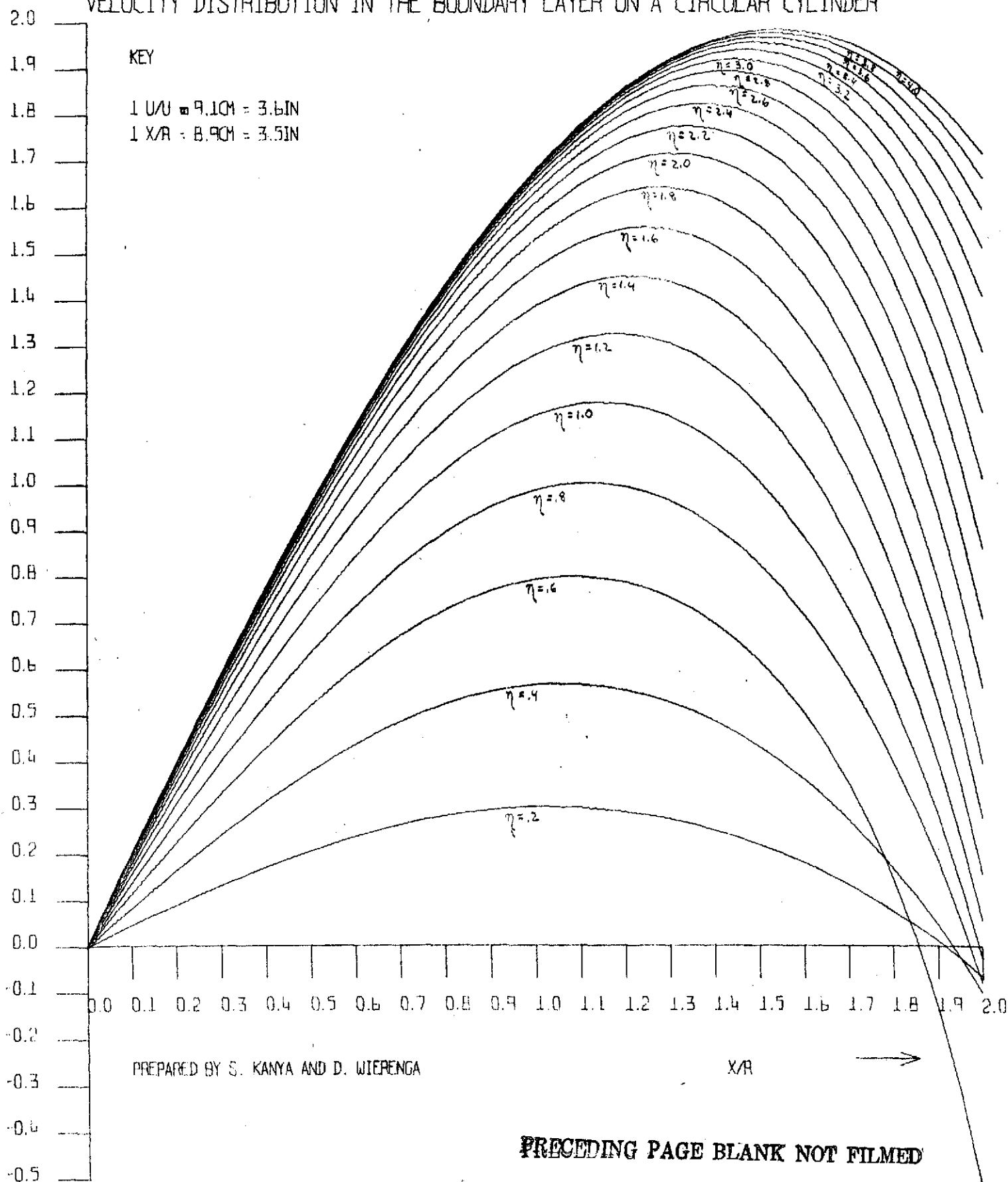
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The First Set of Plots

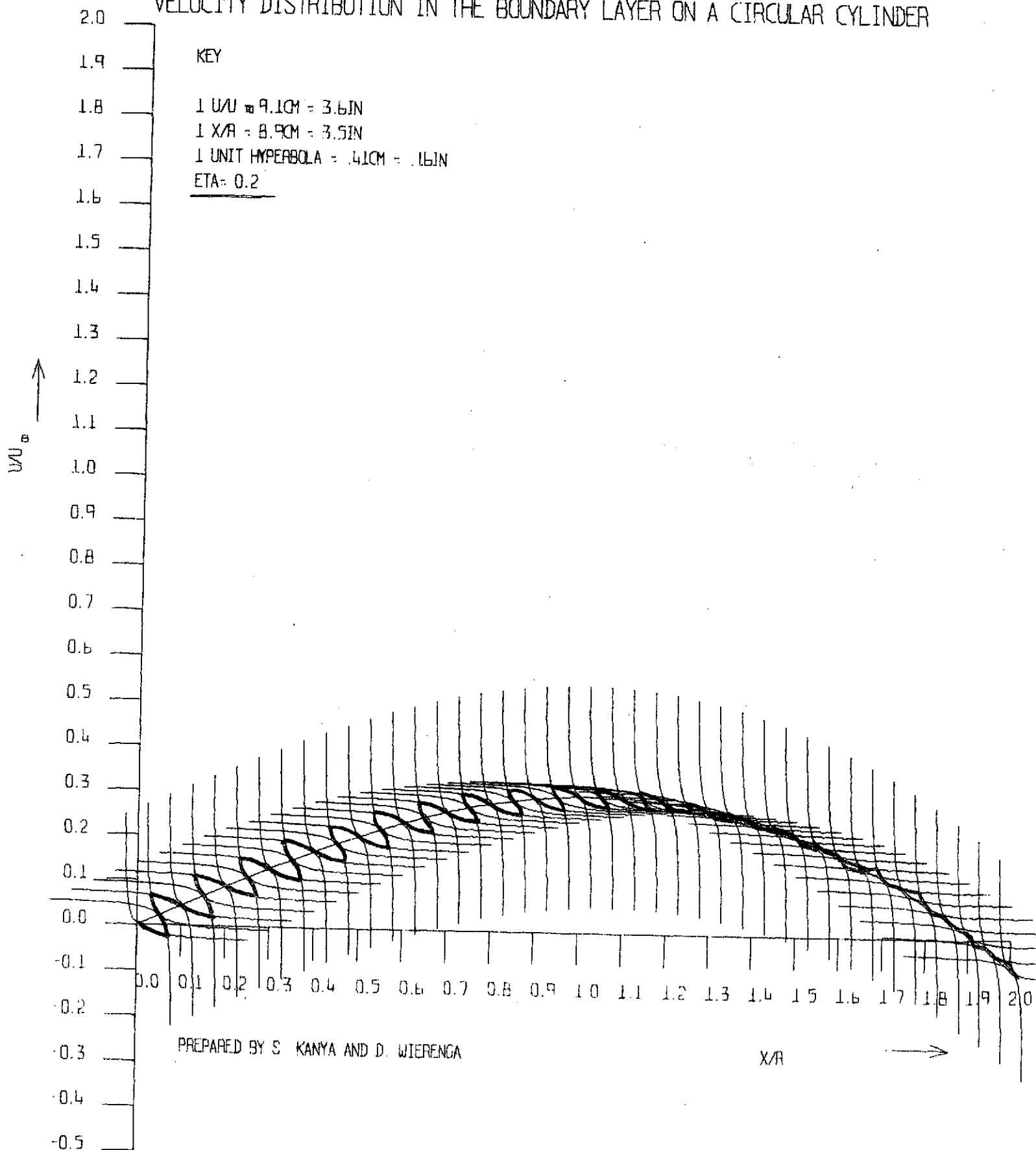
The function $(u U_{\infty}^{-1})$ as the function of $(x R^{-1})$ for various values of the parameter $\eta = 0.2, 0.4, \dots, 4.0$.

No disturbances.

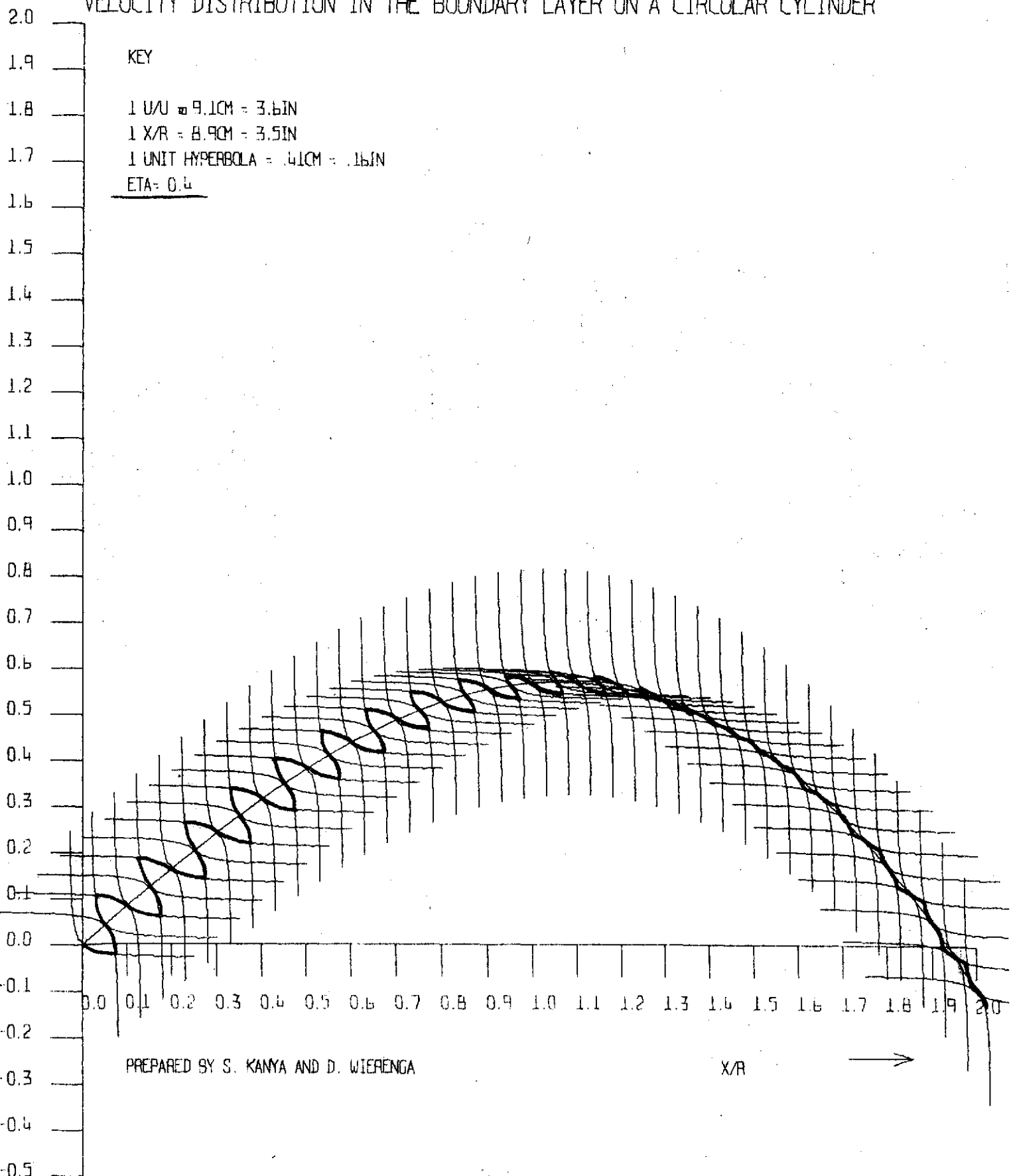
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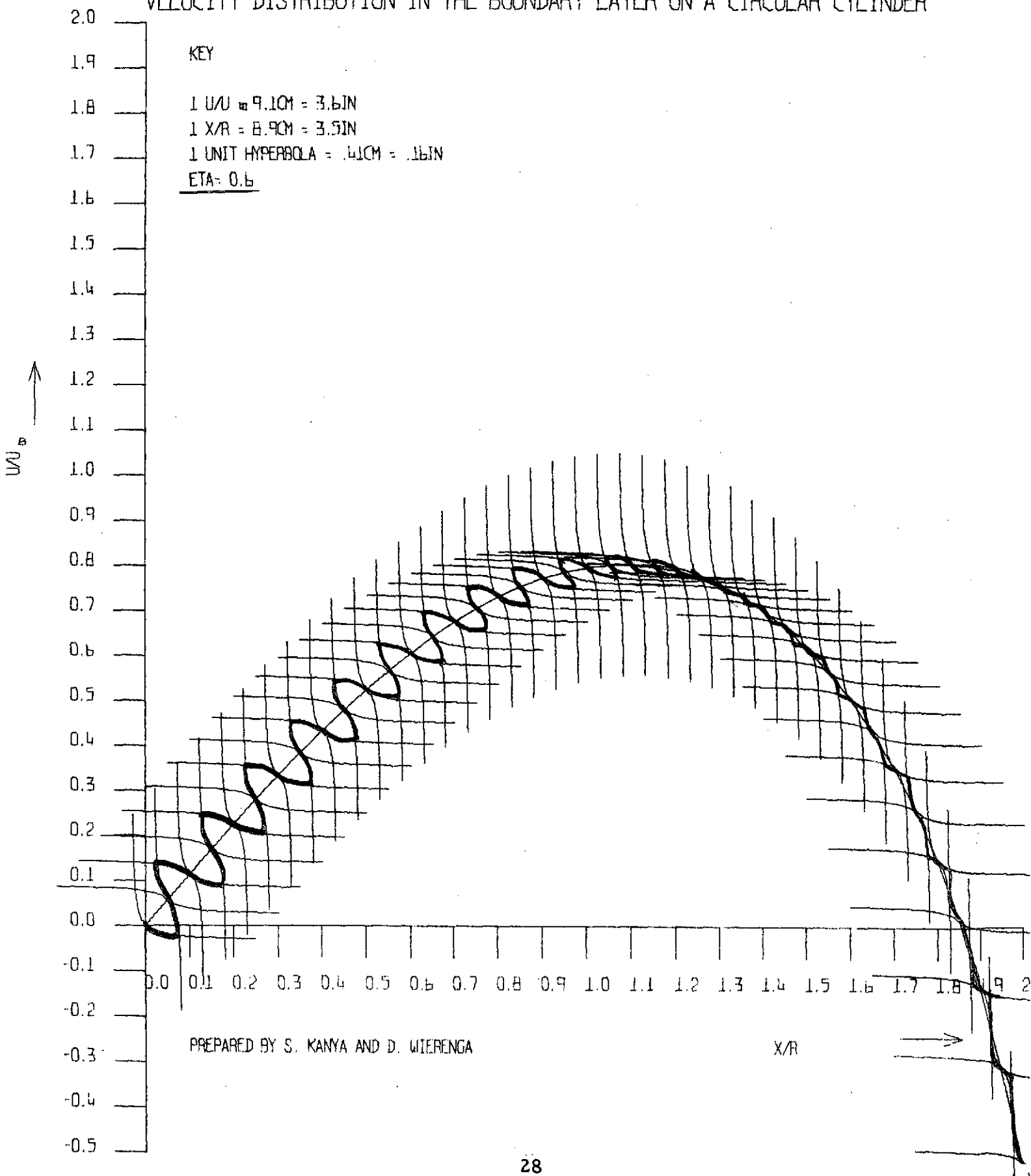
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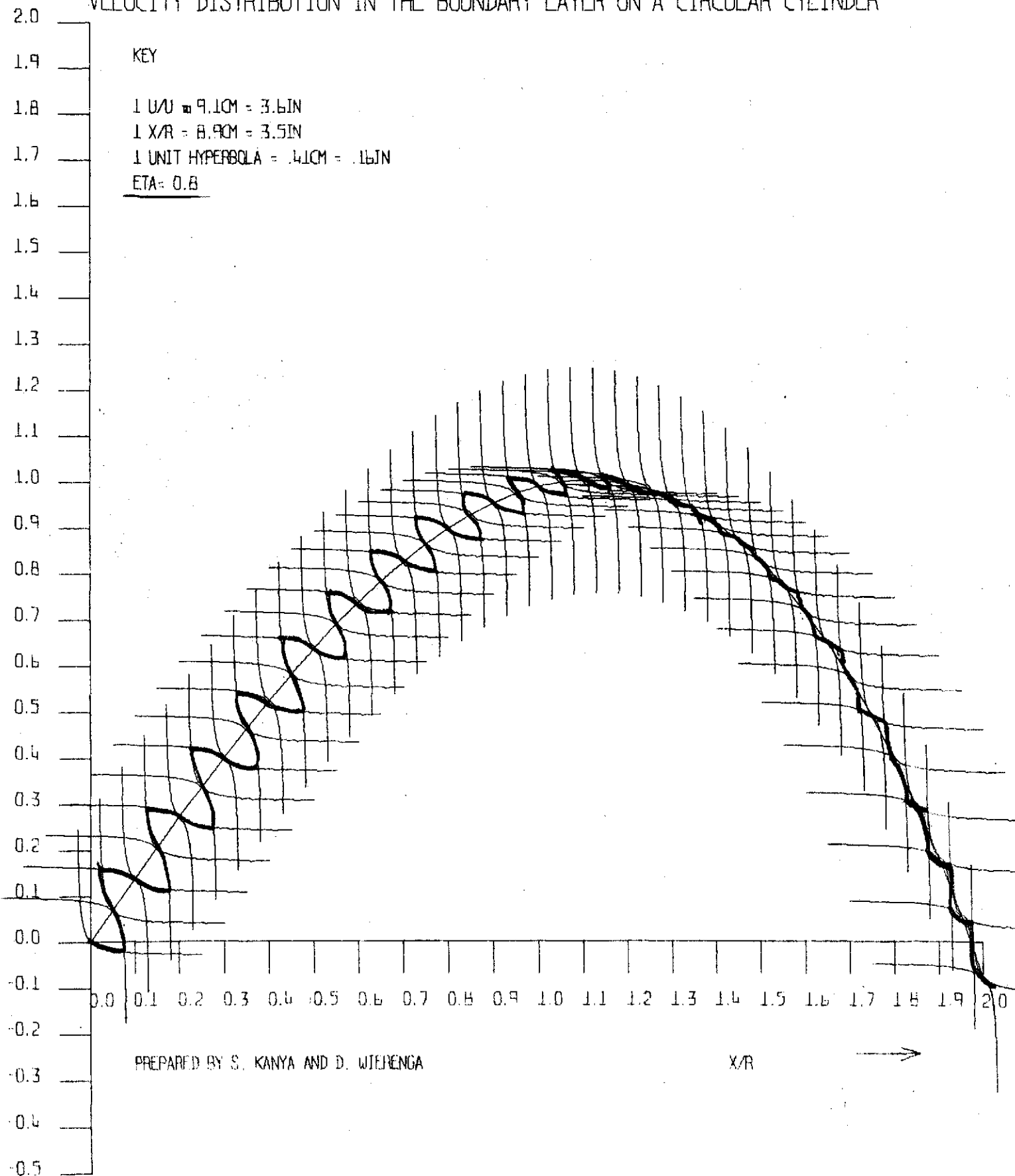
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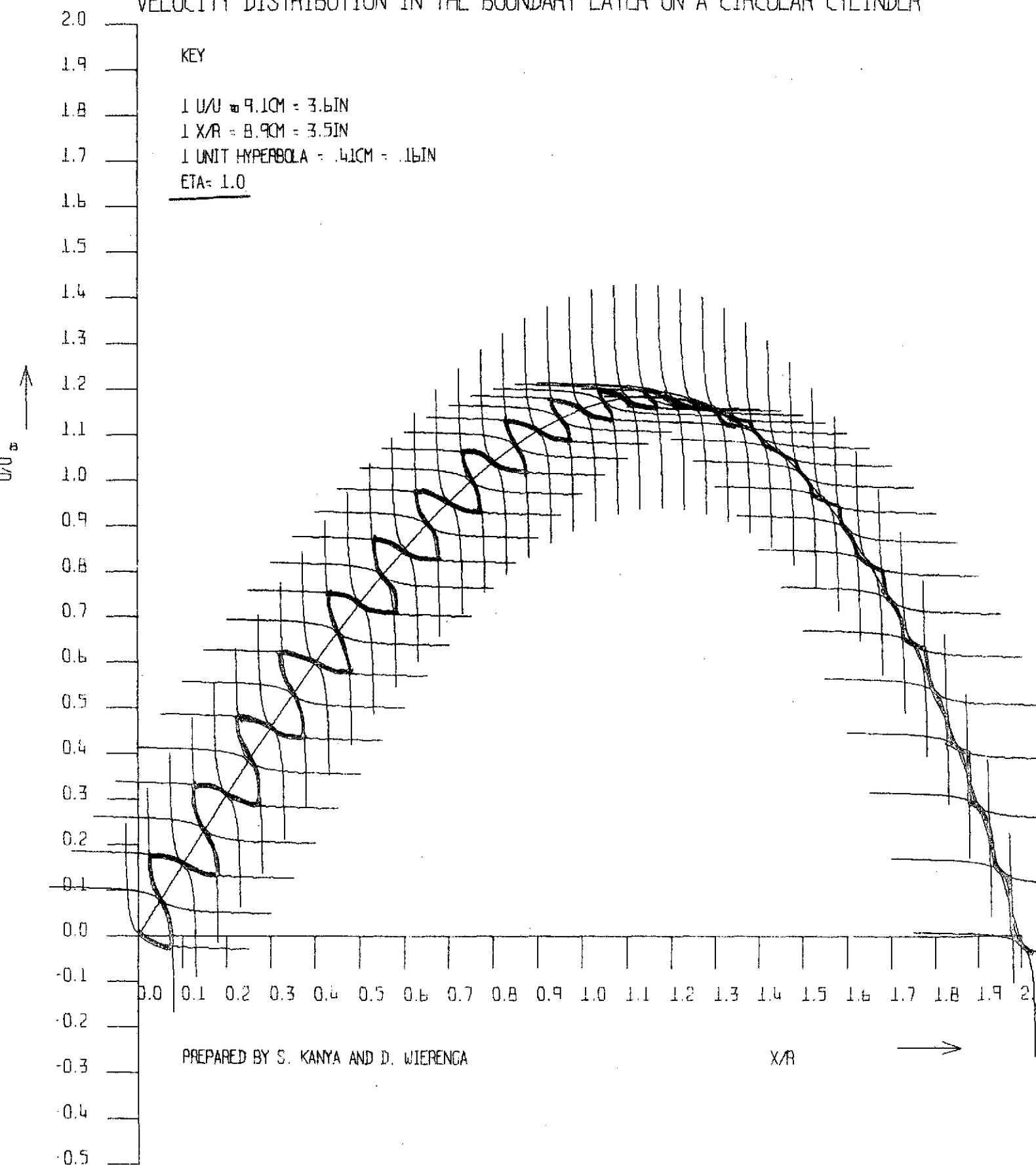
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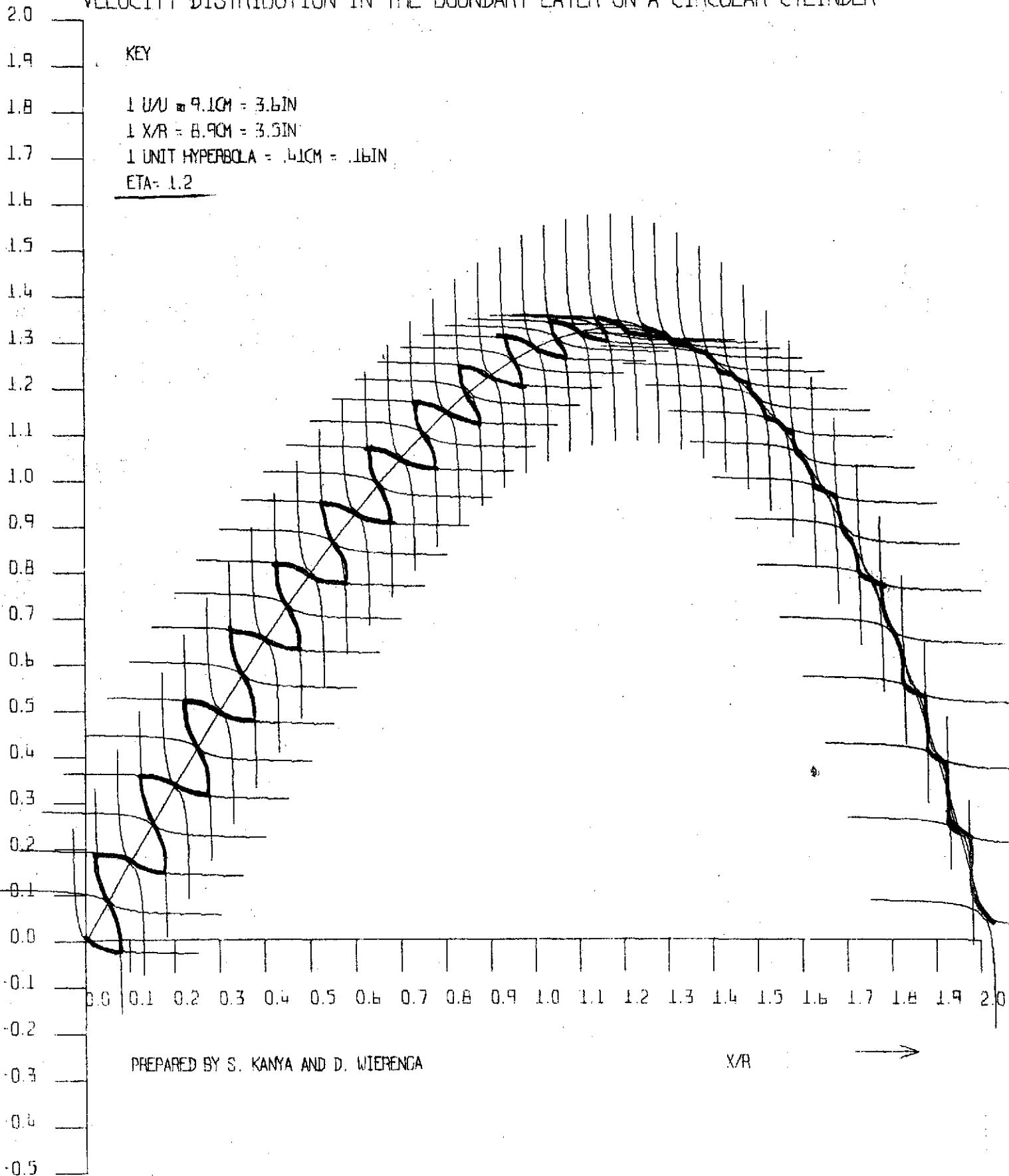
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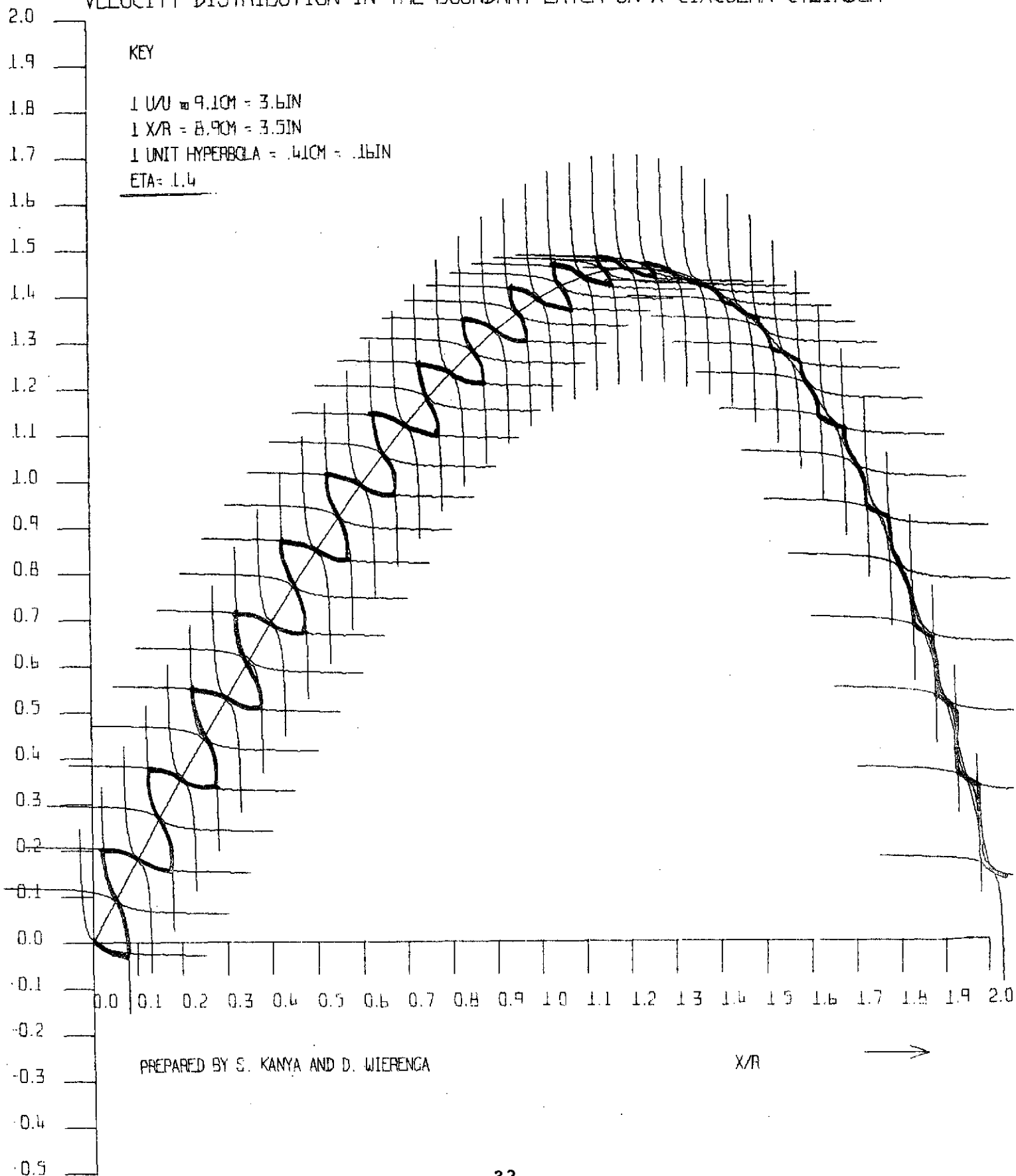
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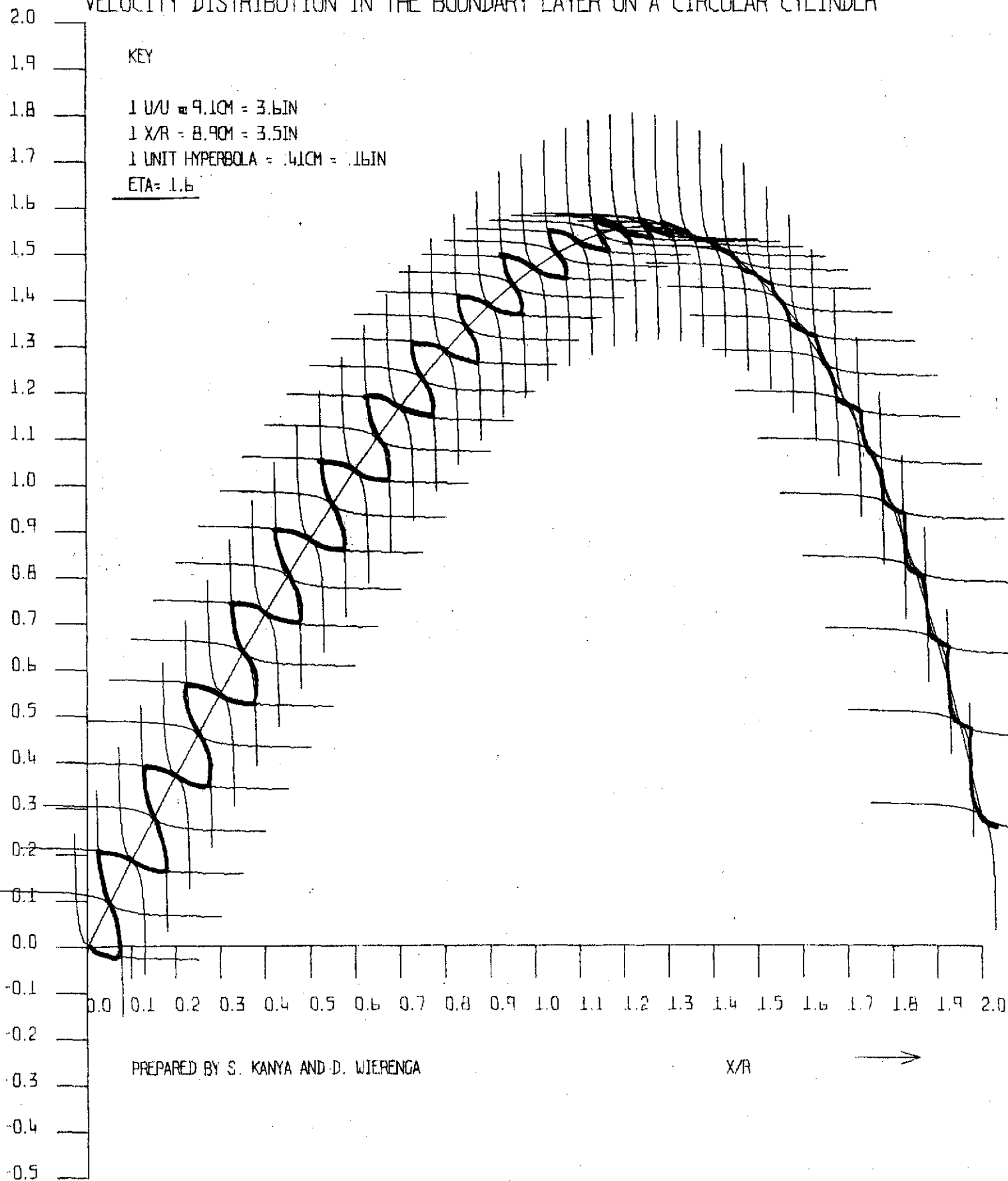
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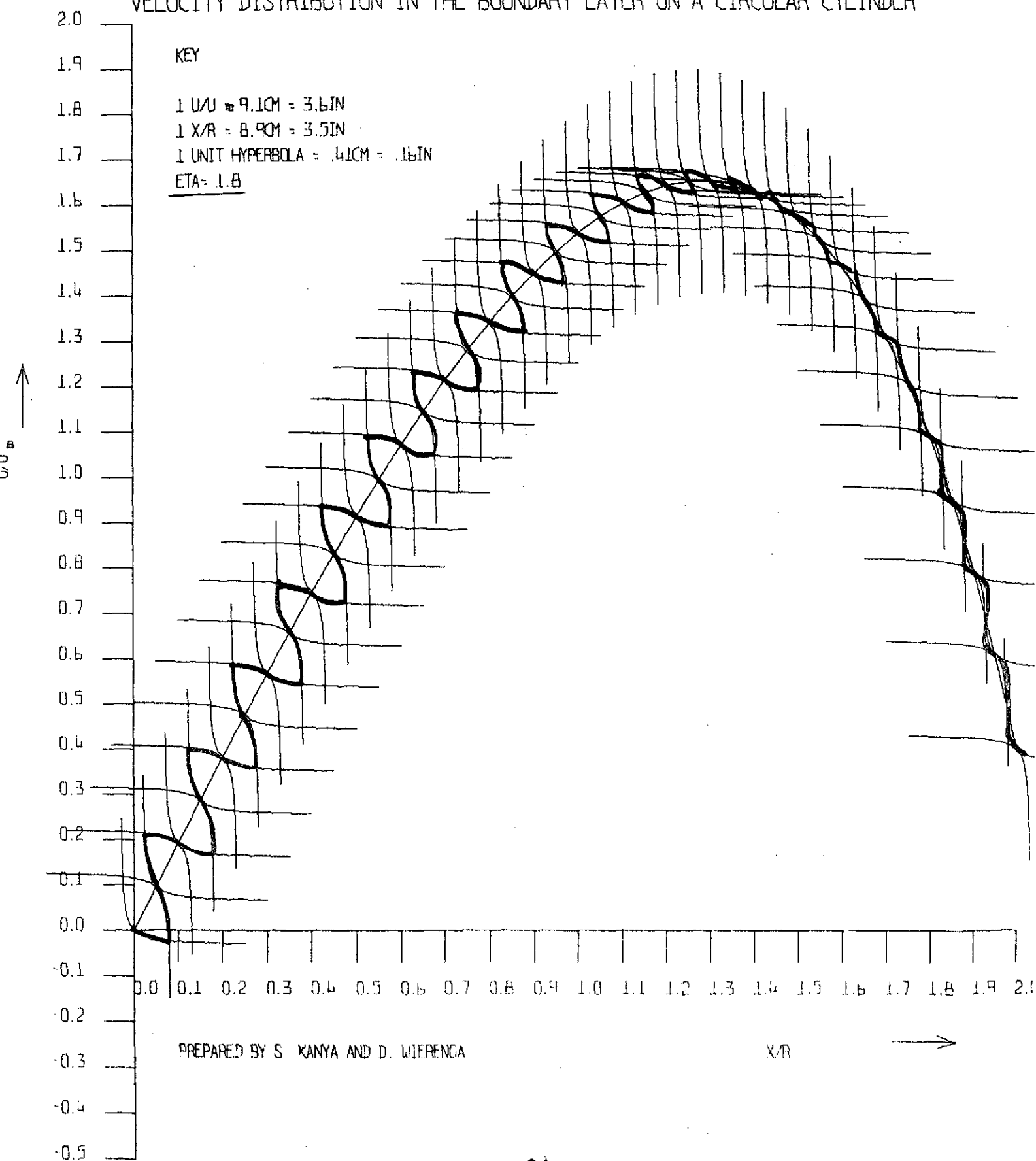
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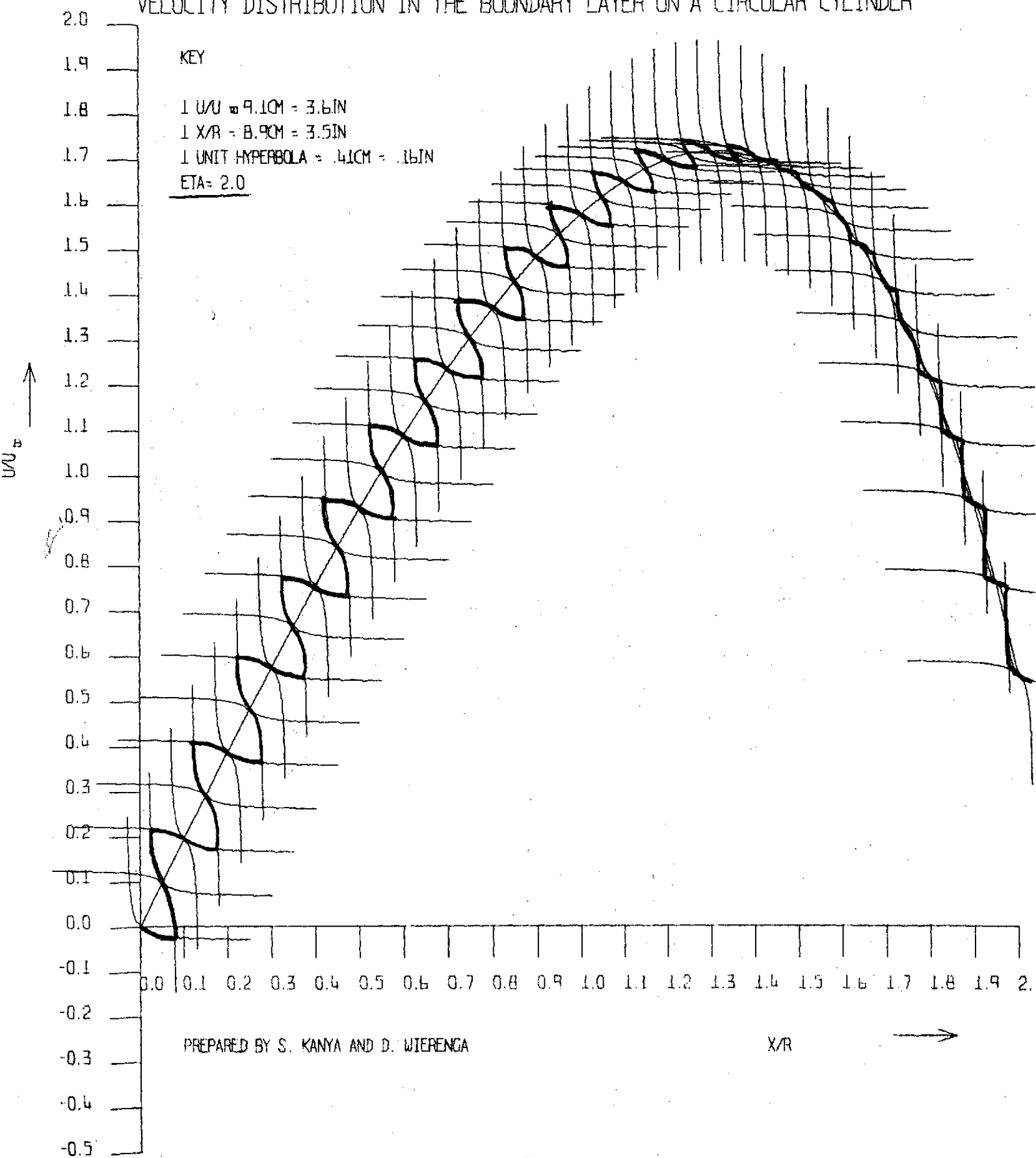
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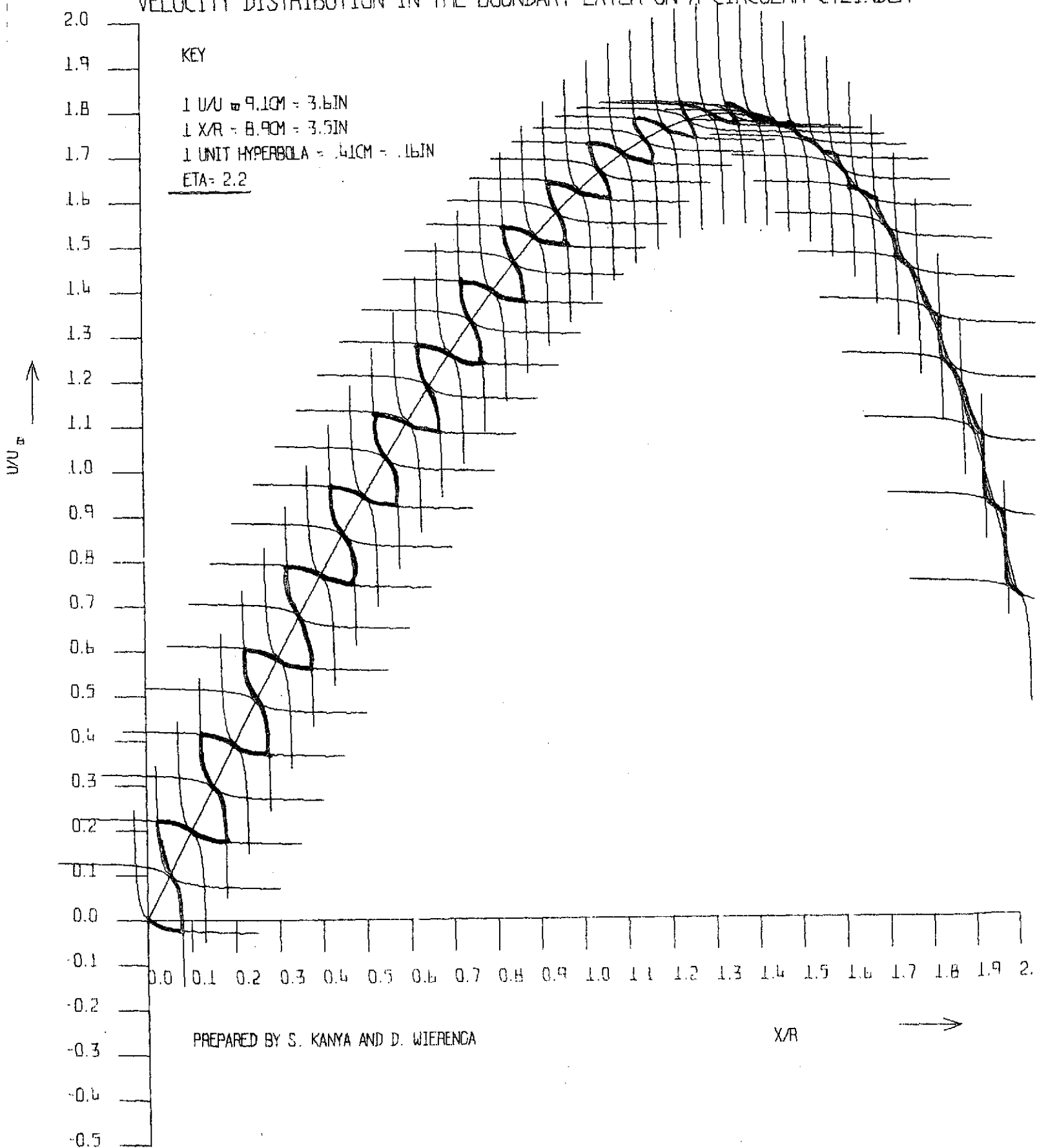
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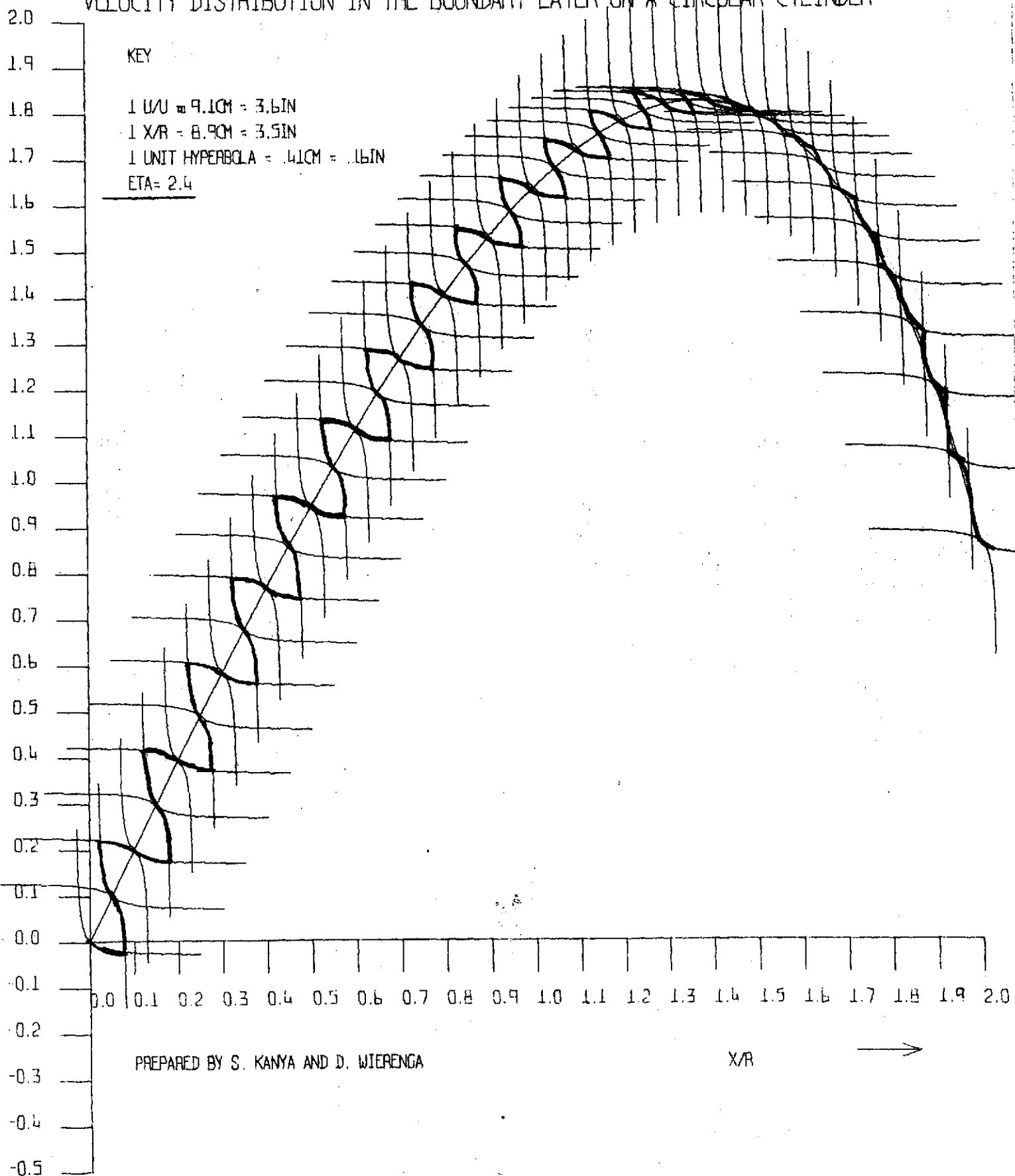
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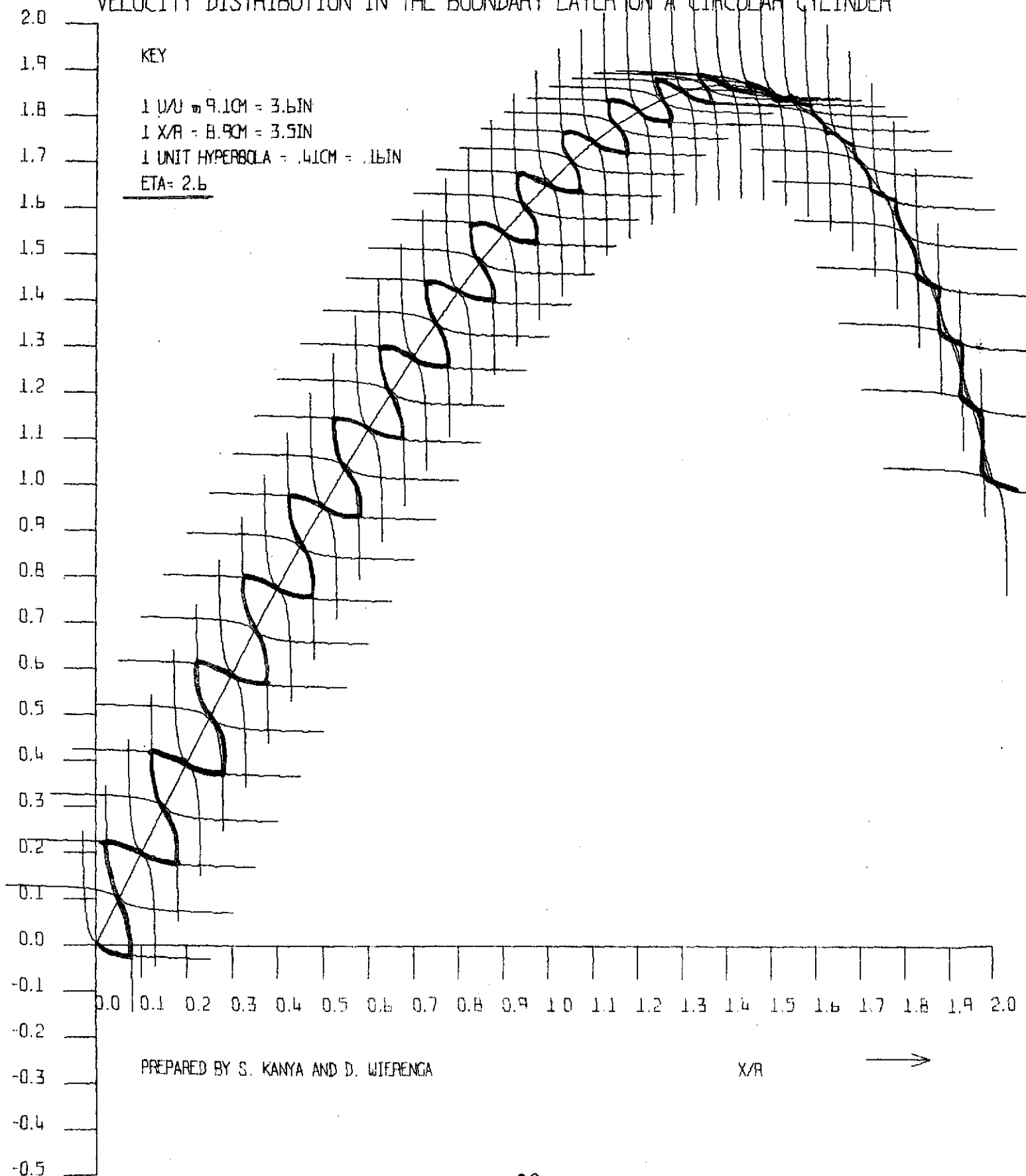
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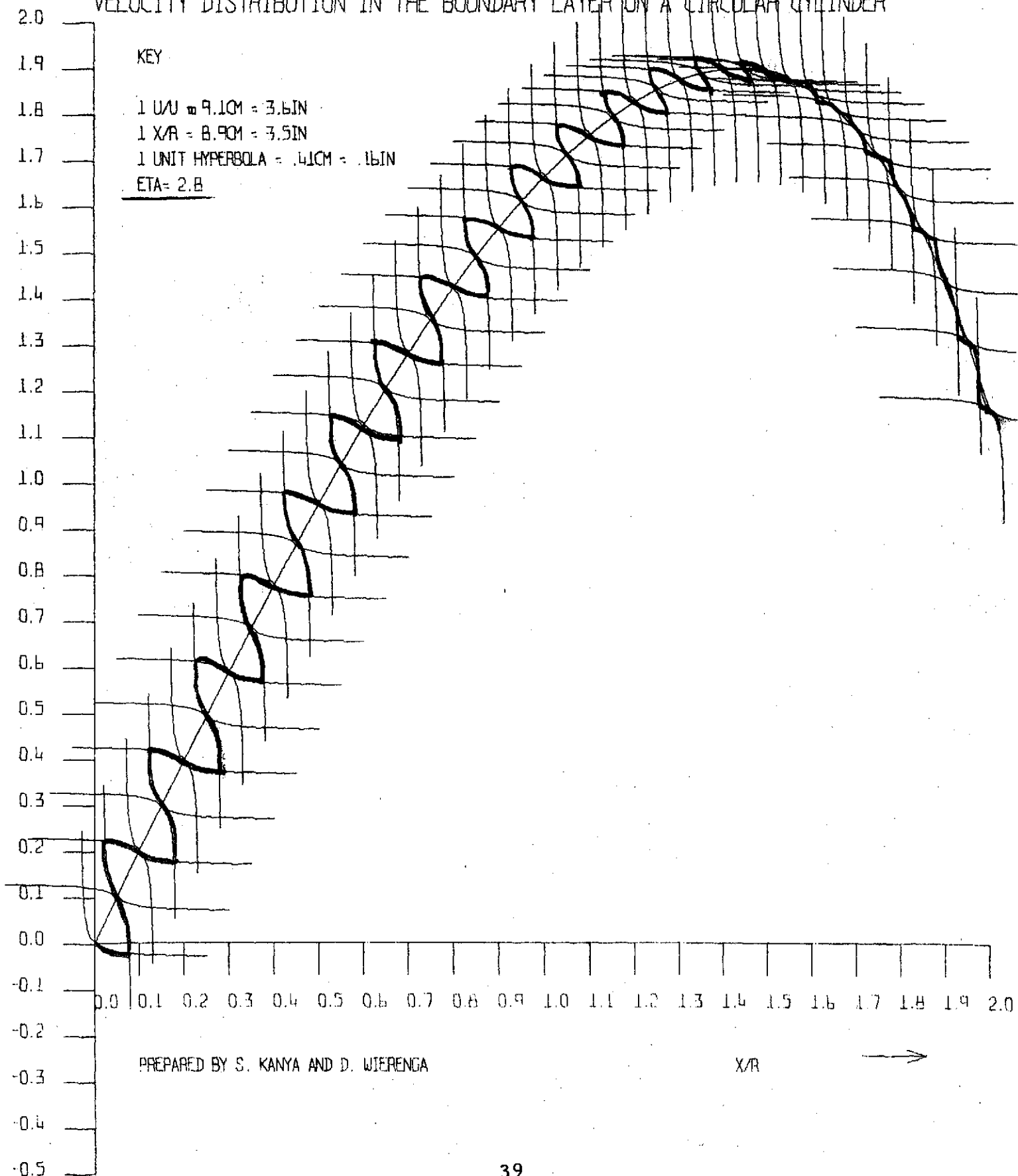
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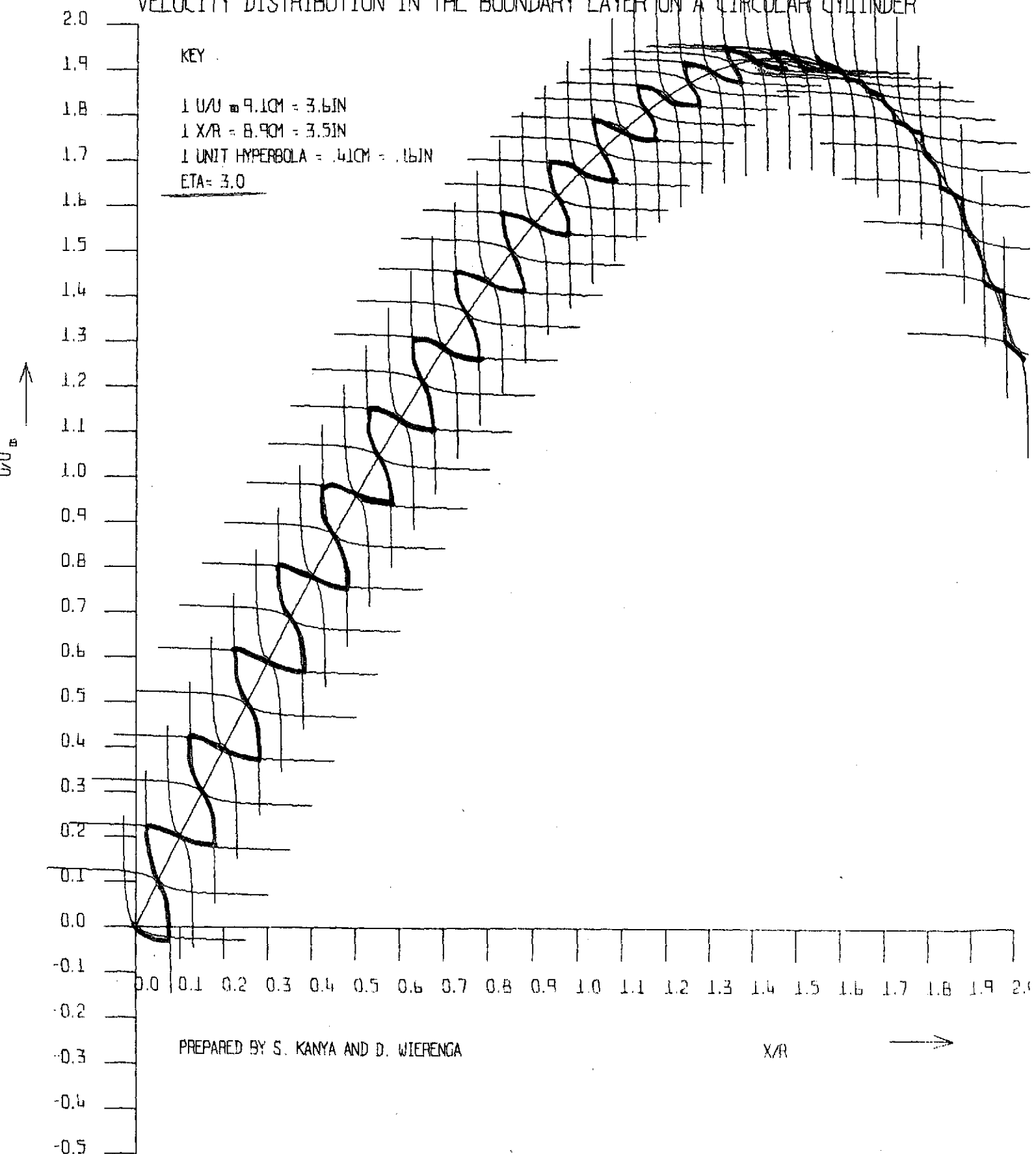
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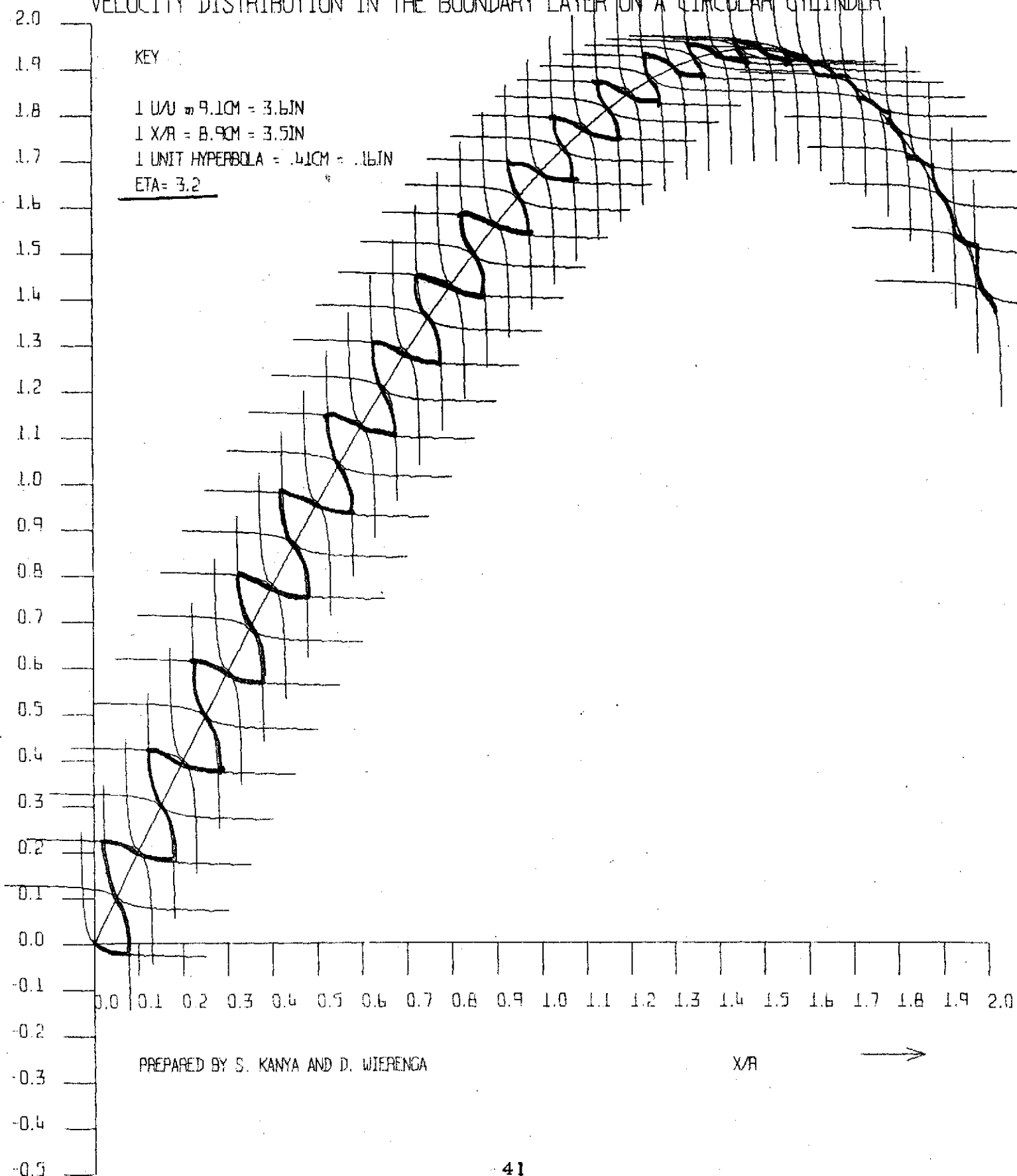
PREPARED BY S. KANYA AND D. WIERENGA

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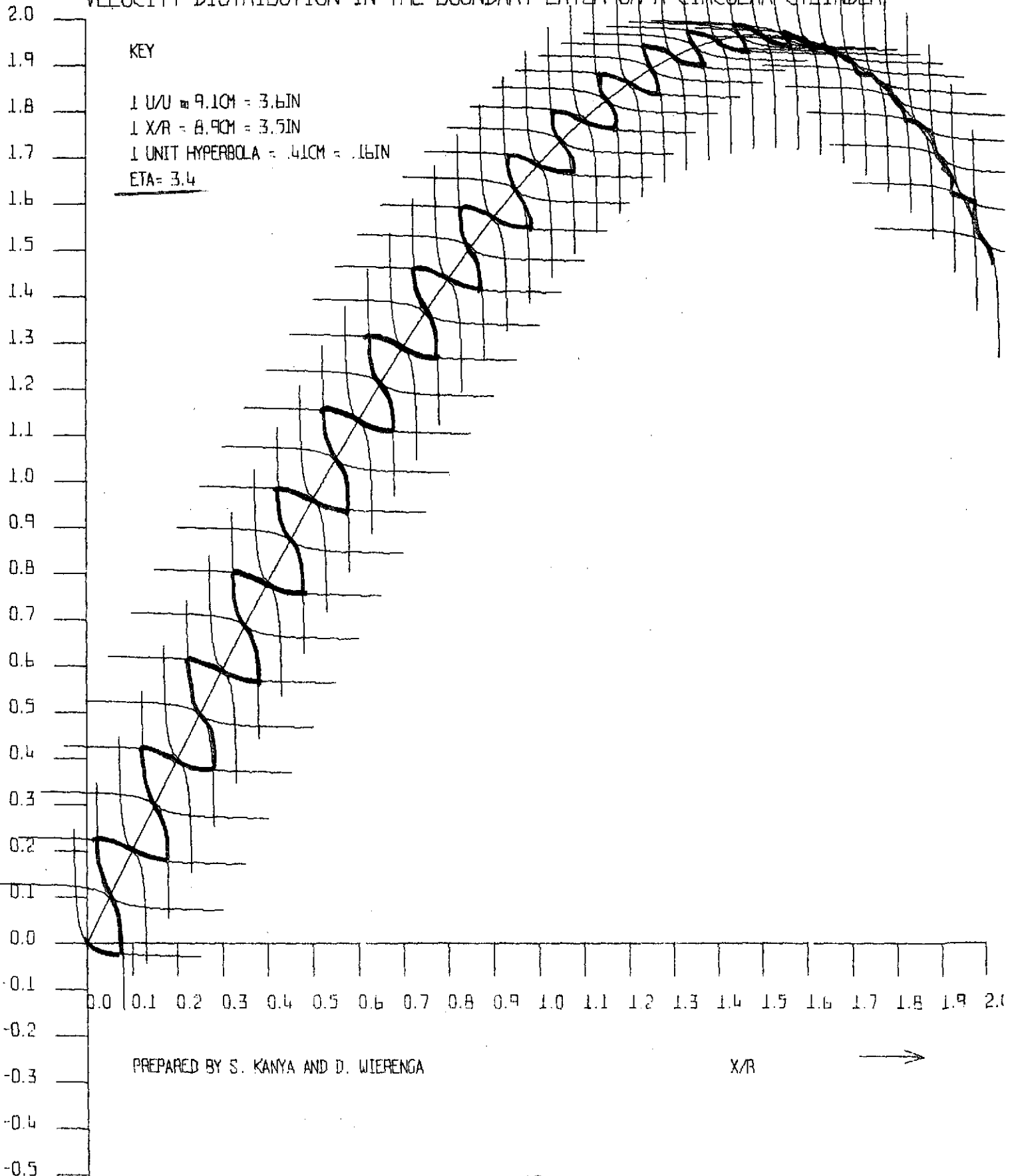
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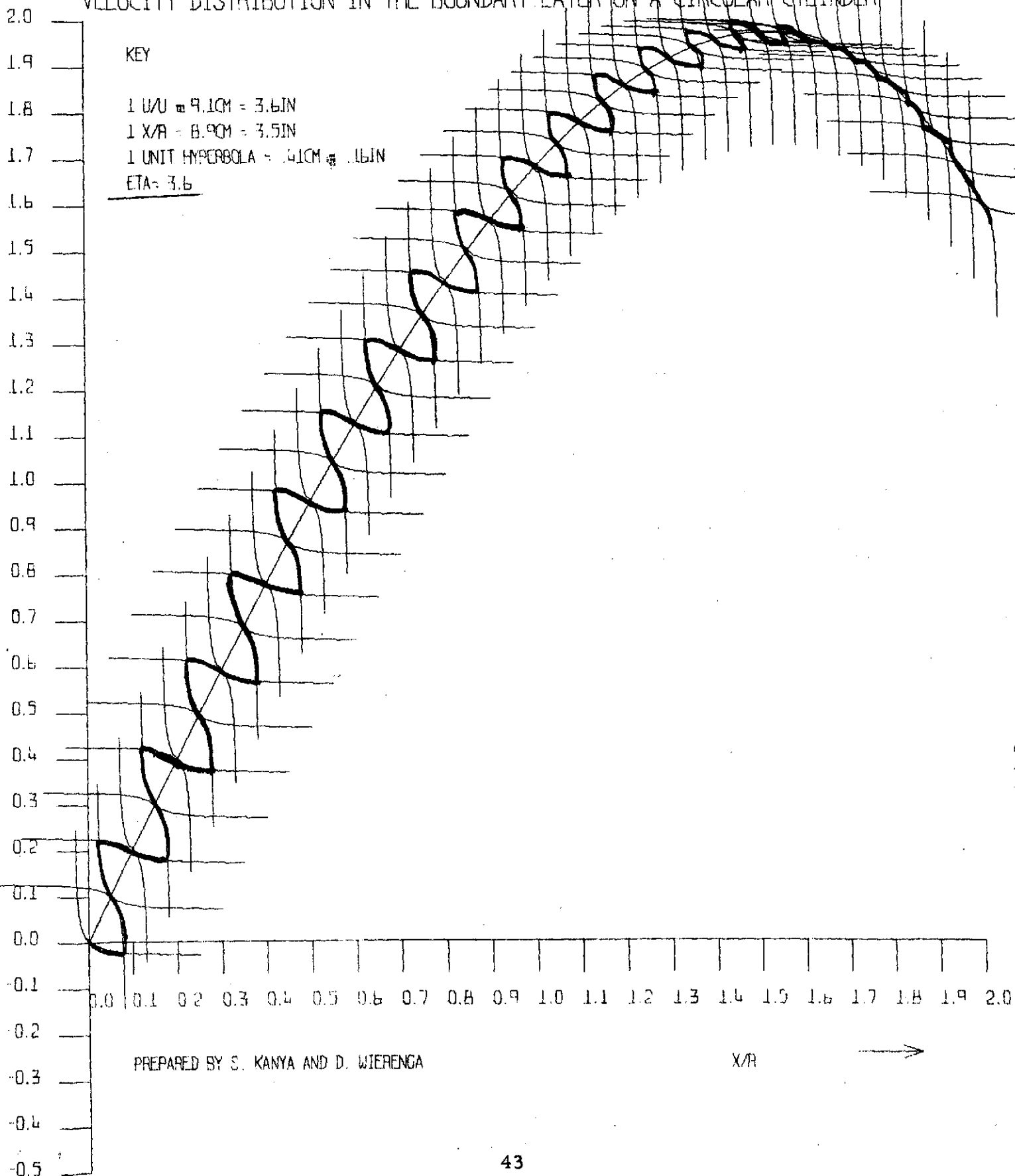
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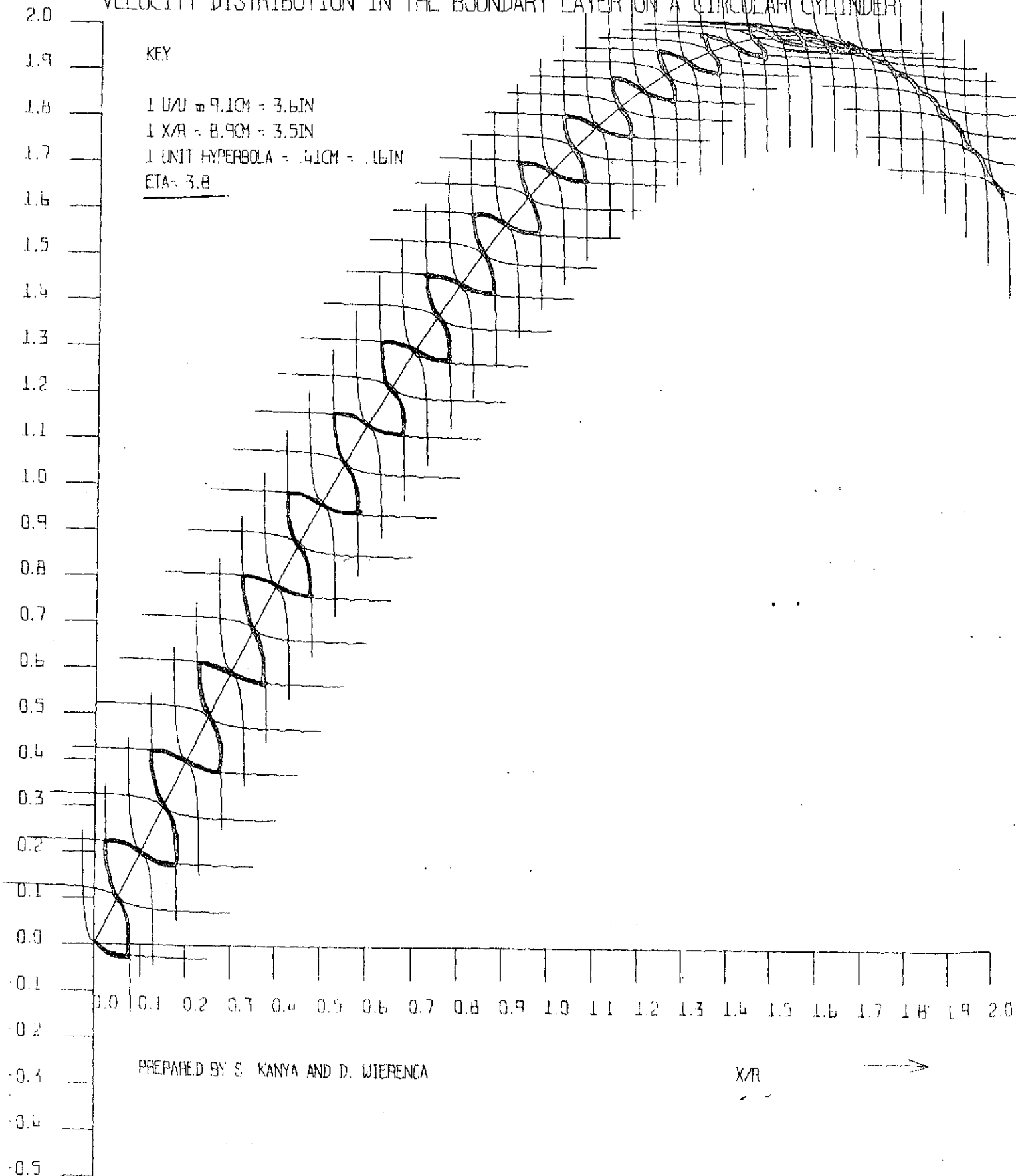
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1 $X/R = 8.9CM = 3.5IN$

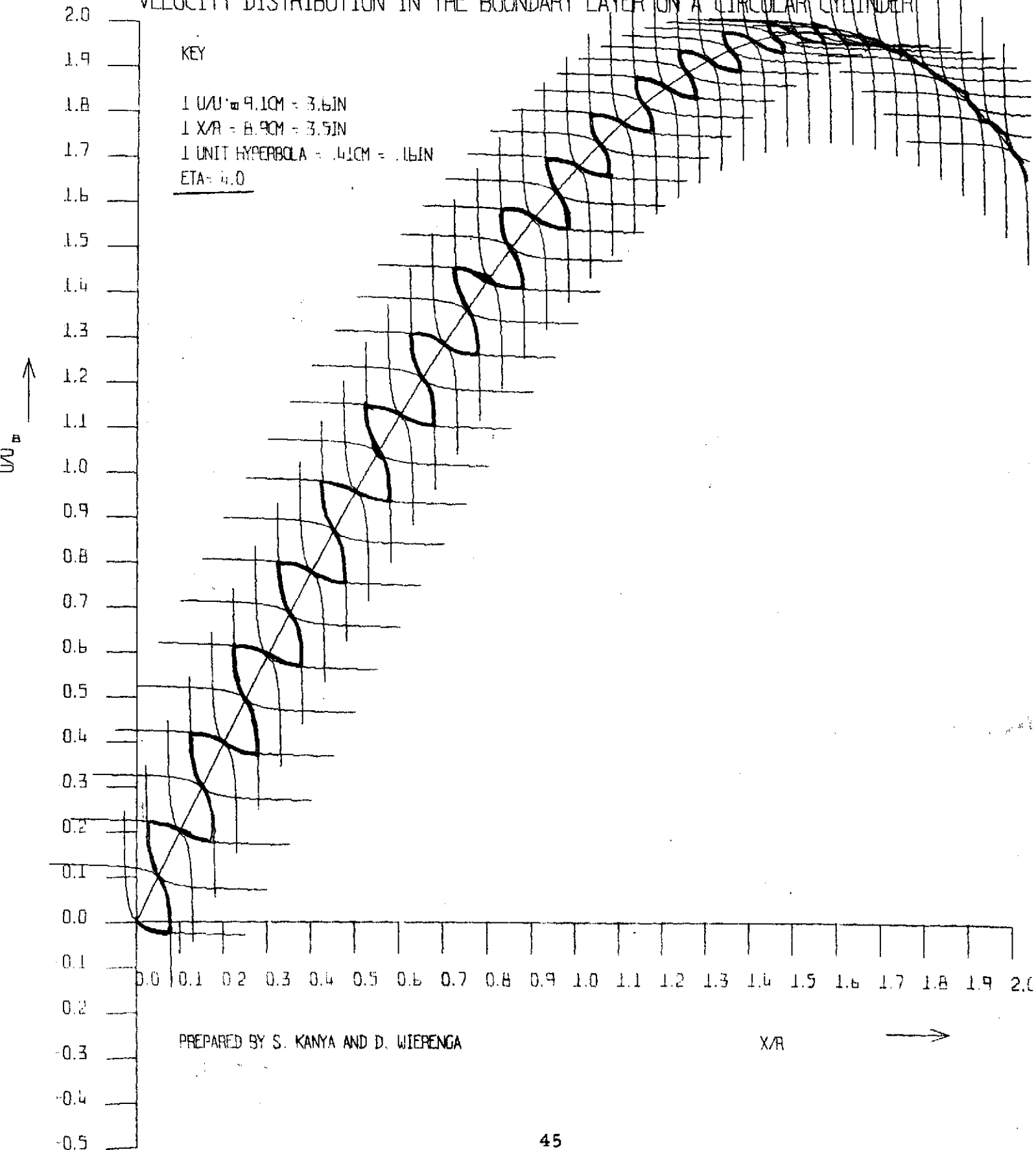
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U/U_∞ ↑



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER



The Second Set of Plots

The function $(u U_{\omega}^{-1} + \omega_z)$ as the function of $(x R^{-1})$ for various values of the parameter $\eta = 0.2, 0.4, \dots, 4.0$. The function $u U_{\omega}^{-1}$ is dimensionless, the function $\omega_z \approx \text{sec}^{-1}$. Curl represents the disturbances superimposed upon the function $u U_{\omega}^{-1}$. Since the dimensionless velocity and the disturbance have different dimensions, one has to assume that the plots are performed in time-frozen conditions.

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VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

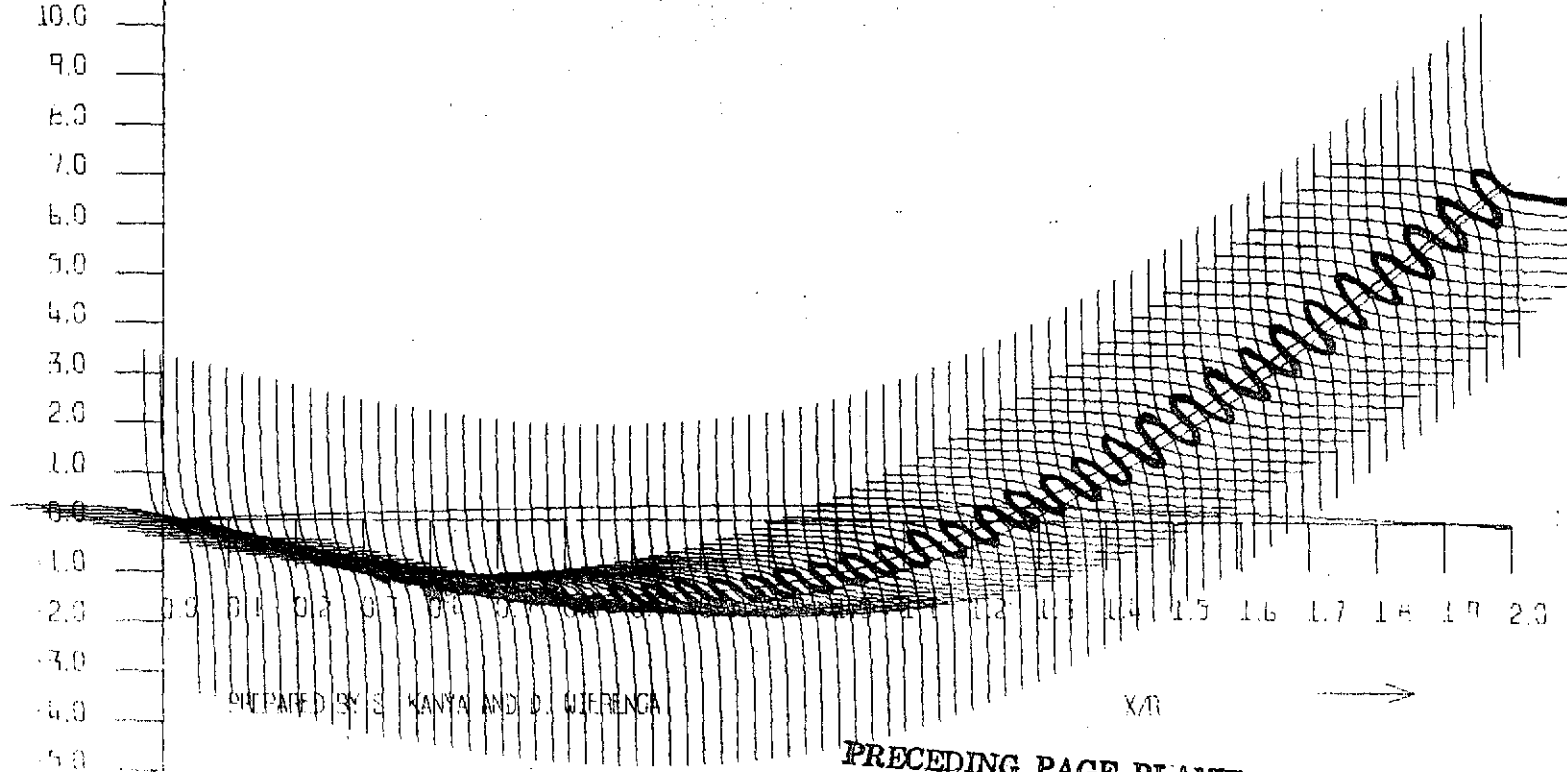
KEY

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1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .61CM = .16IN

ETA = 0.2



PREPARED BY S. KANYA AND D. WIERENGA

X/R

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$u, v, w + \text{DISTURBANCE}$

VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

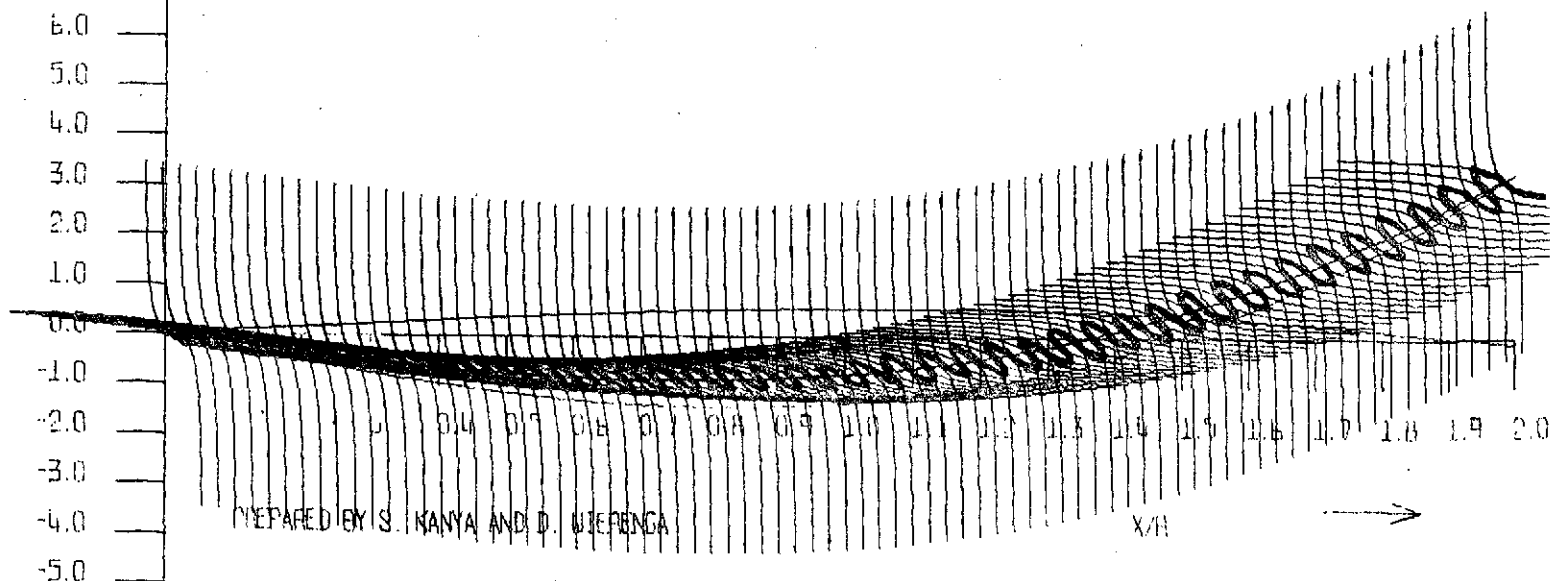
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VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

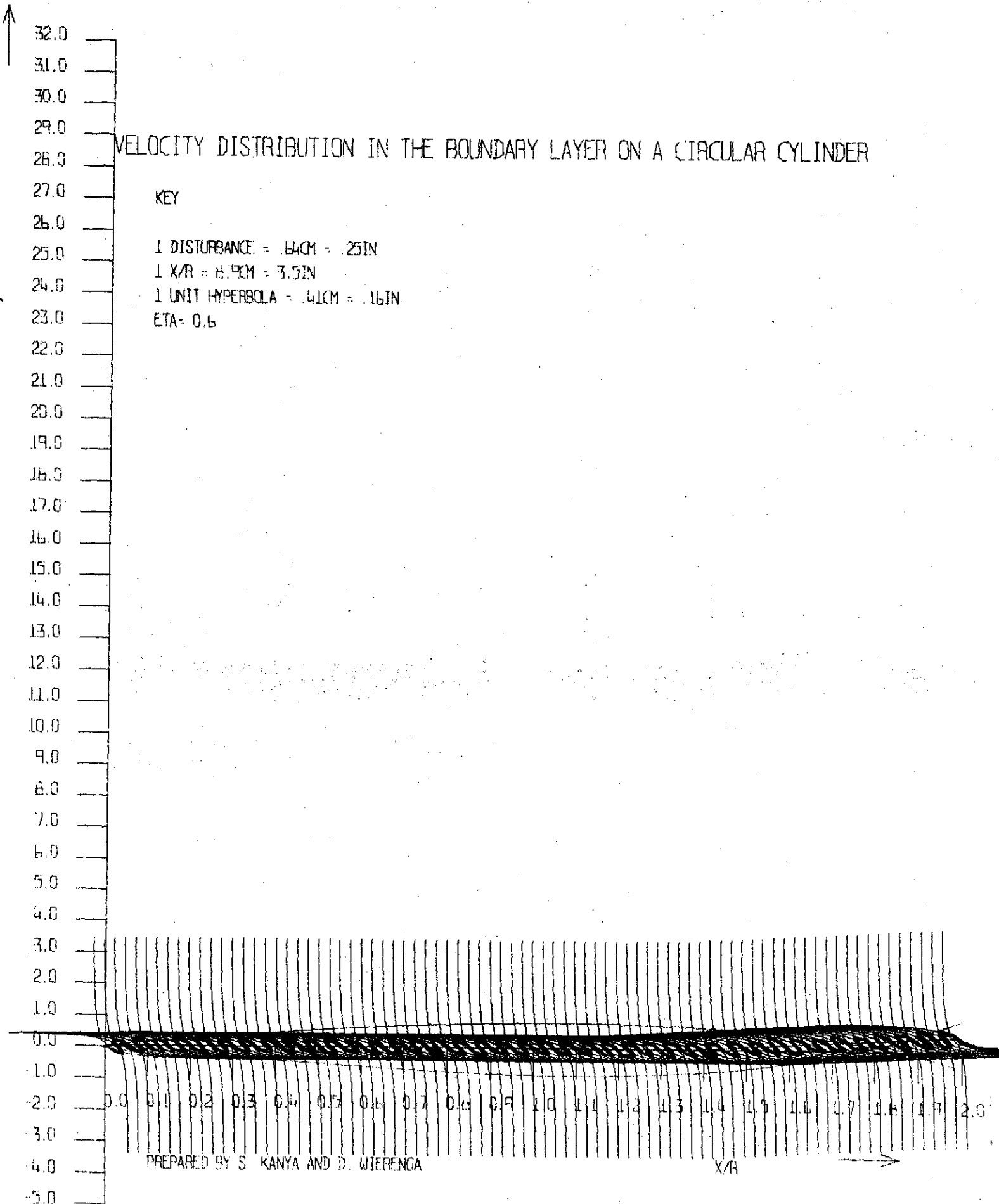
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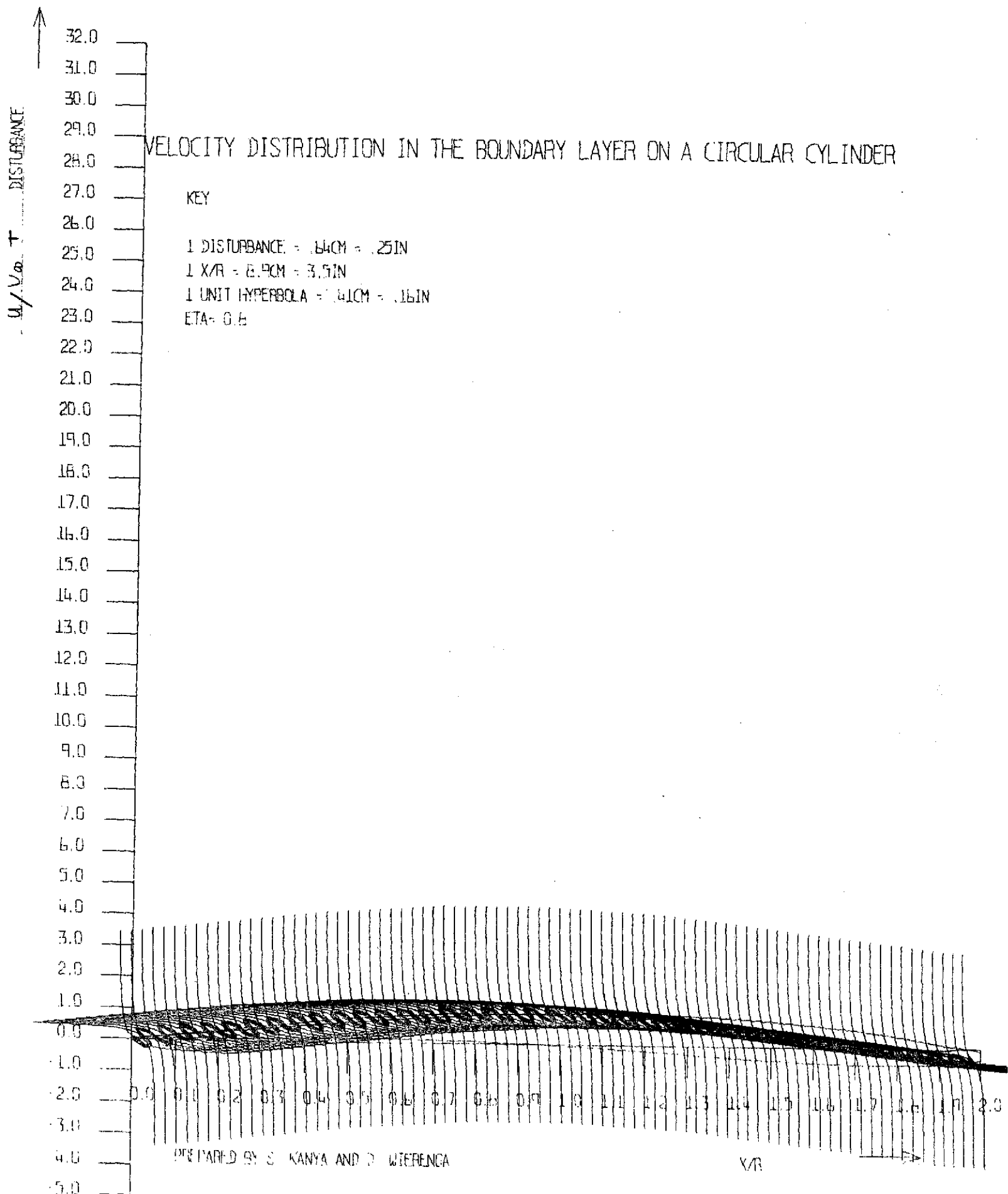
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1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 0.6





VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

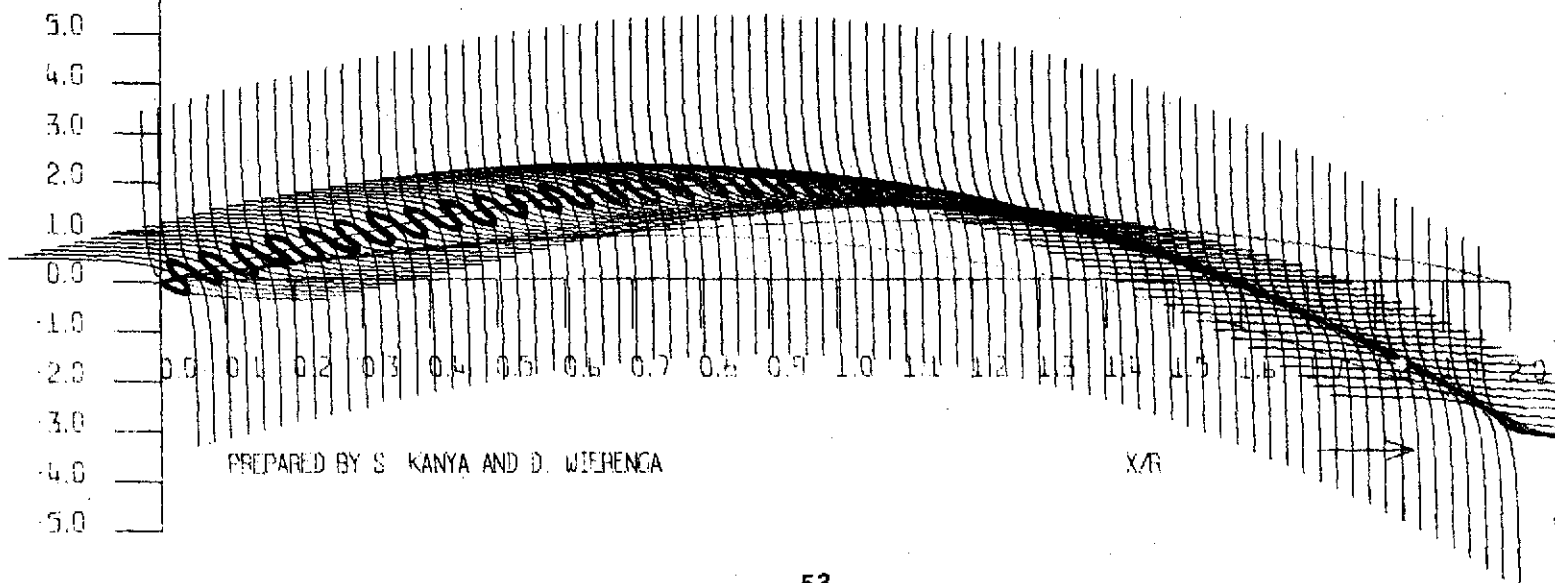
KEY

1 DISTURBANCE = $6.4\text{CM} = .25\text{IN}$

1 $X/R = 6.9\text{CM} = 3.5\text{IN}$

1 UNIT HYPERBOLA = $.4\text{CM} = .16\text{IN}$

$\text{ETA} = 1.0$



up to DISTANCE

VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

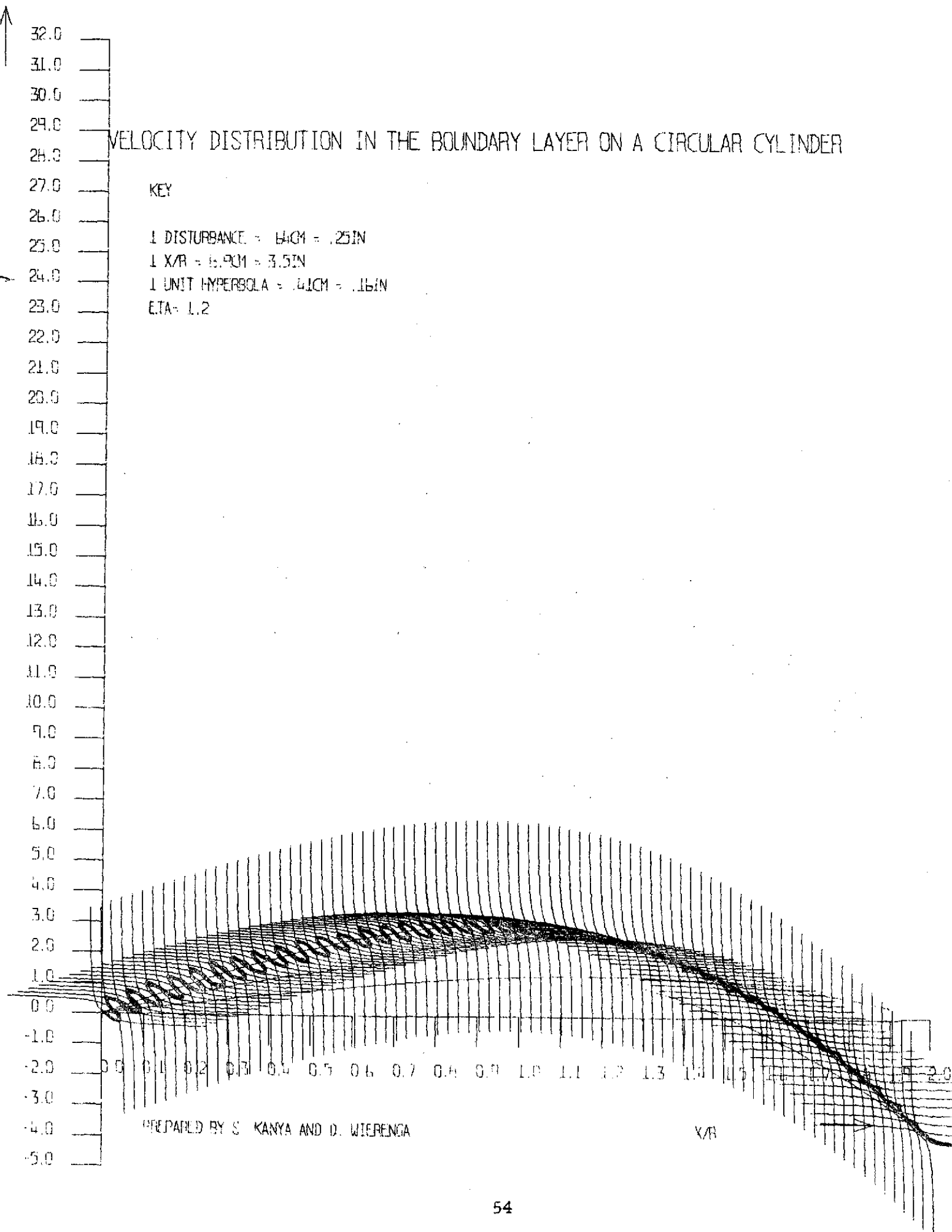
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1 $X/R = 6.9CM = 3.5IN$

1 UNIT HYPERBOLA = $14CM = .16IN$

$\epsilon = 1.2$



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

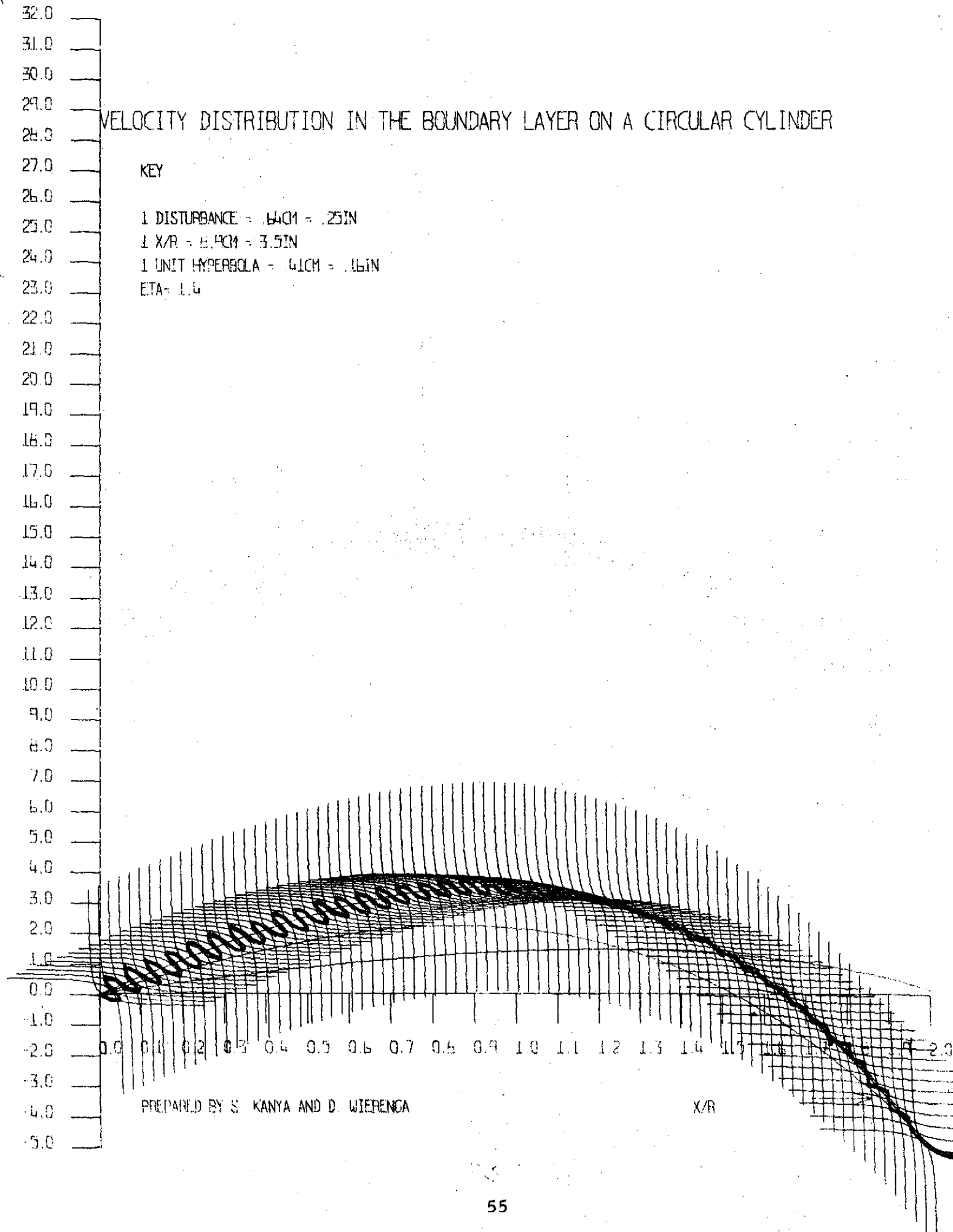
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1 X/R = .64CM = 3.5IN

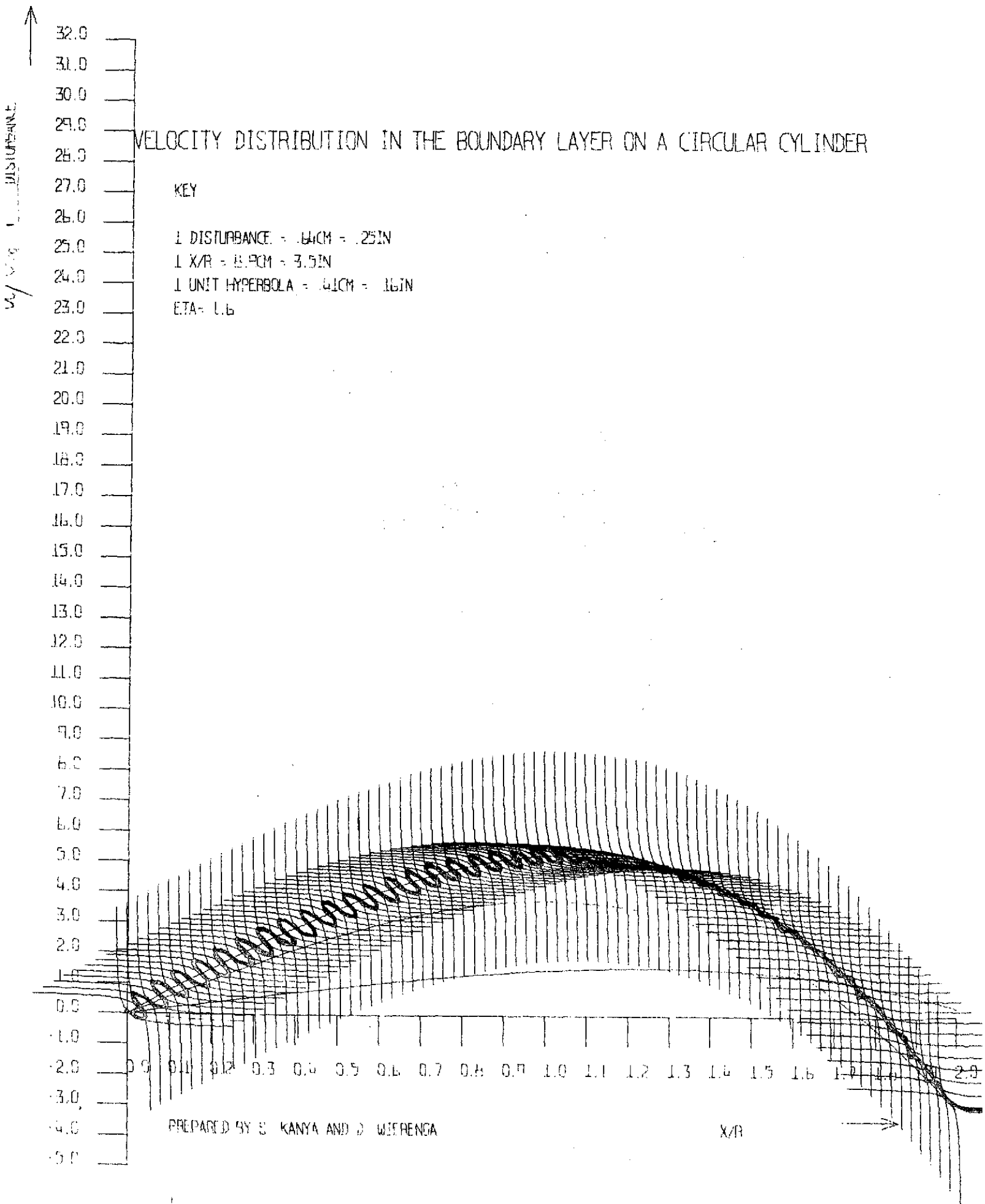
1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 1.6



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X/R



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

KEY

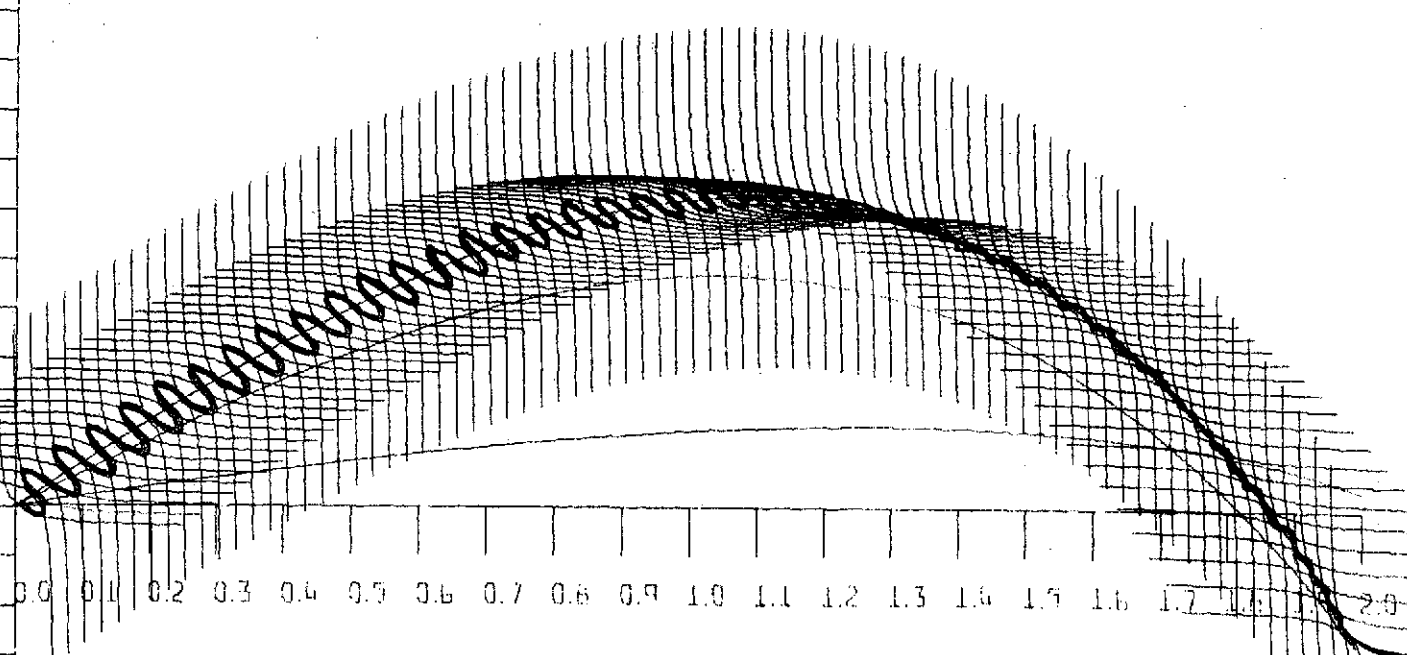
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1 X/R = 6.9CM = 3.5IN

1 UNIT HYPERBOLA = .41CM = .16IN

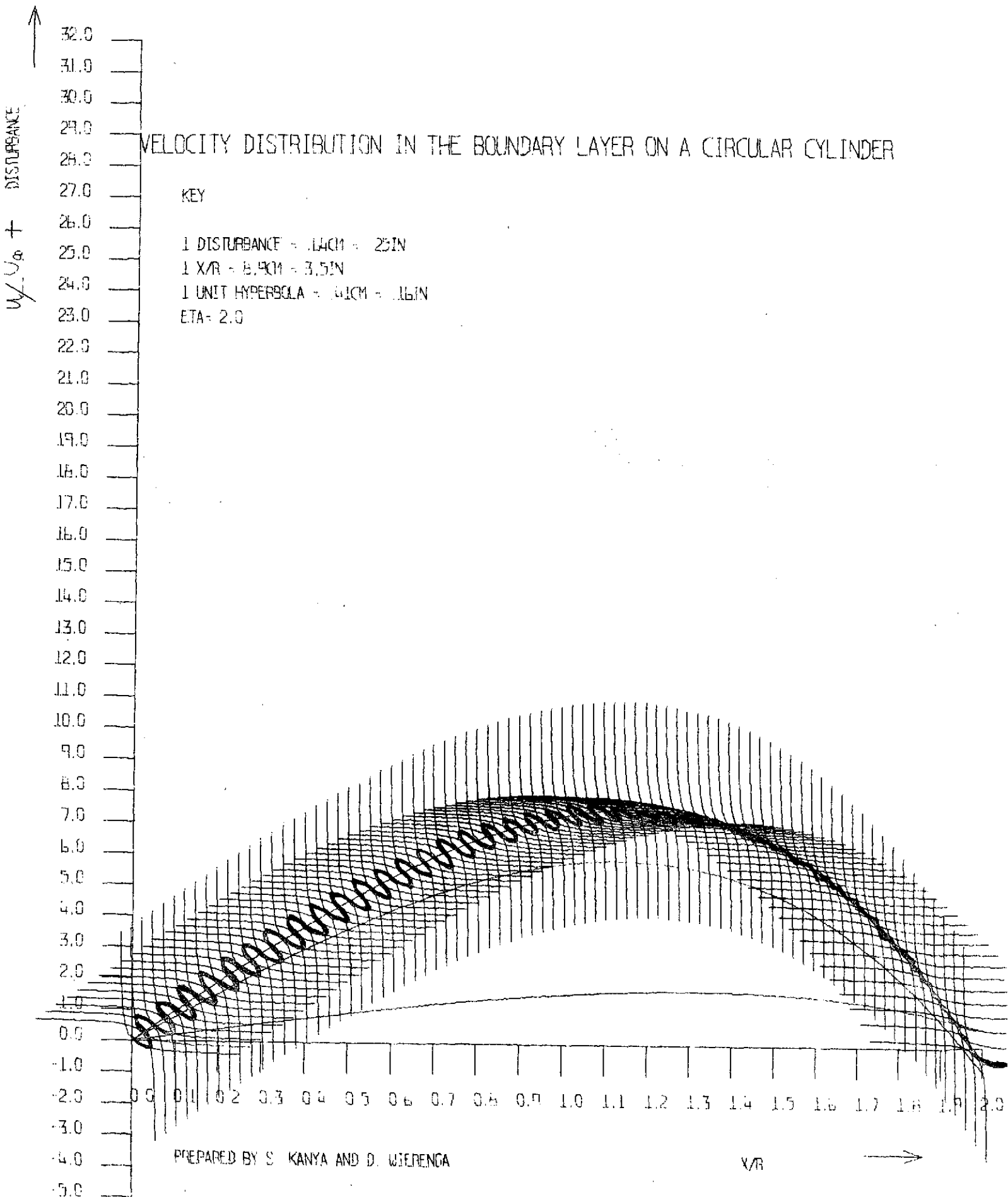
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29.0
28.0
27.0
26.0
25.0
24.0
23.0
22.0
21.0
20.0
19.0
18.0
17.0
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0



PREPARED BY S. KANYA AND D. WIERENGA

X/R



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

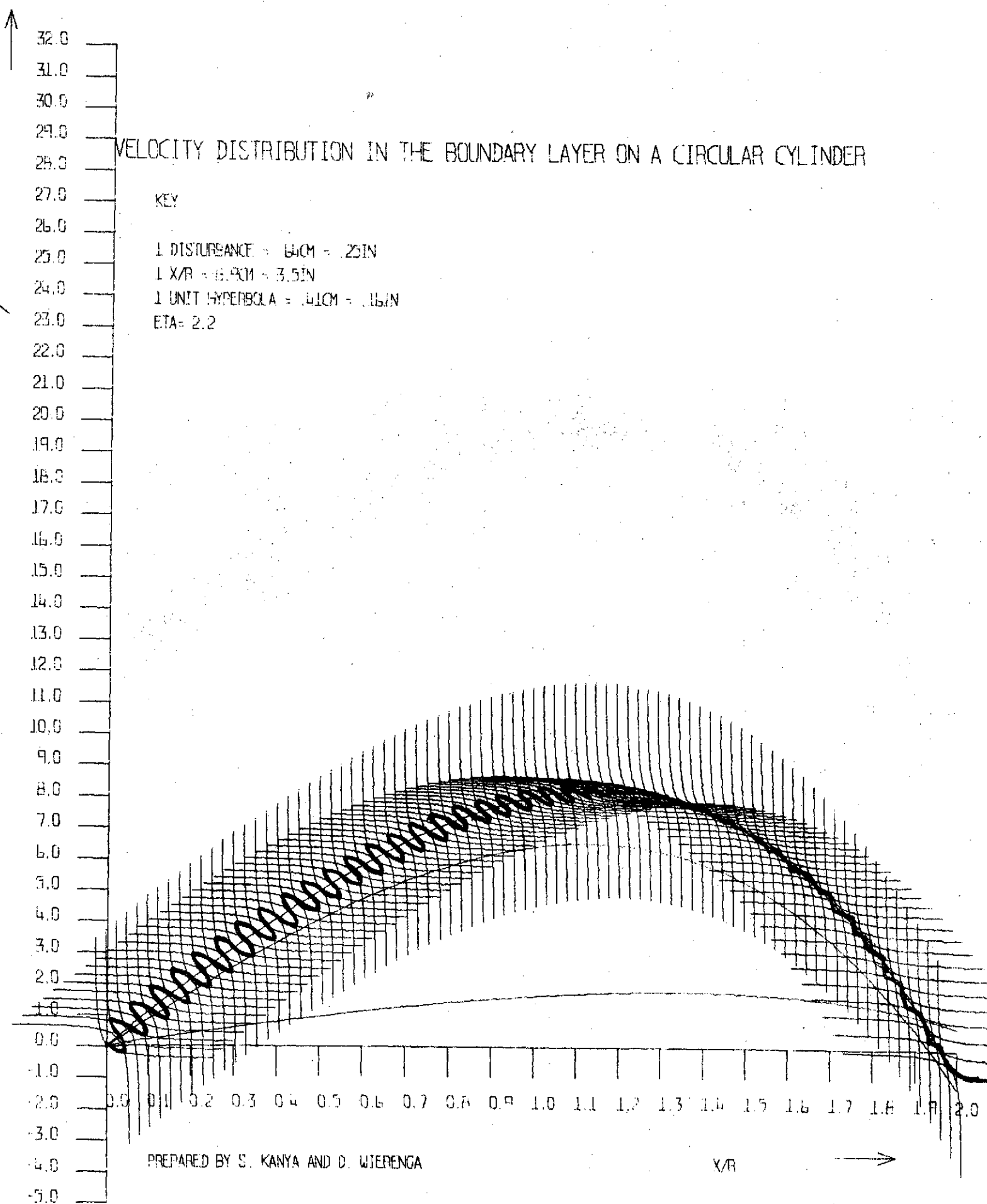
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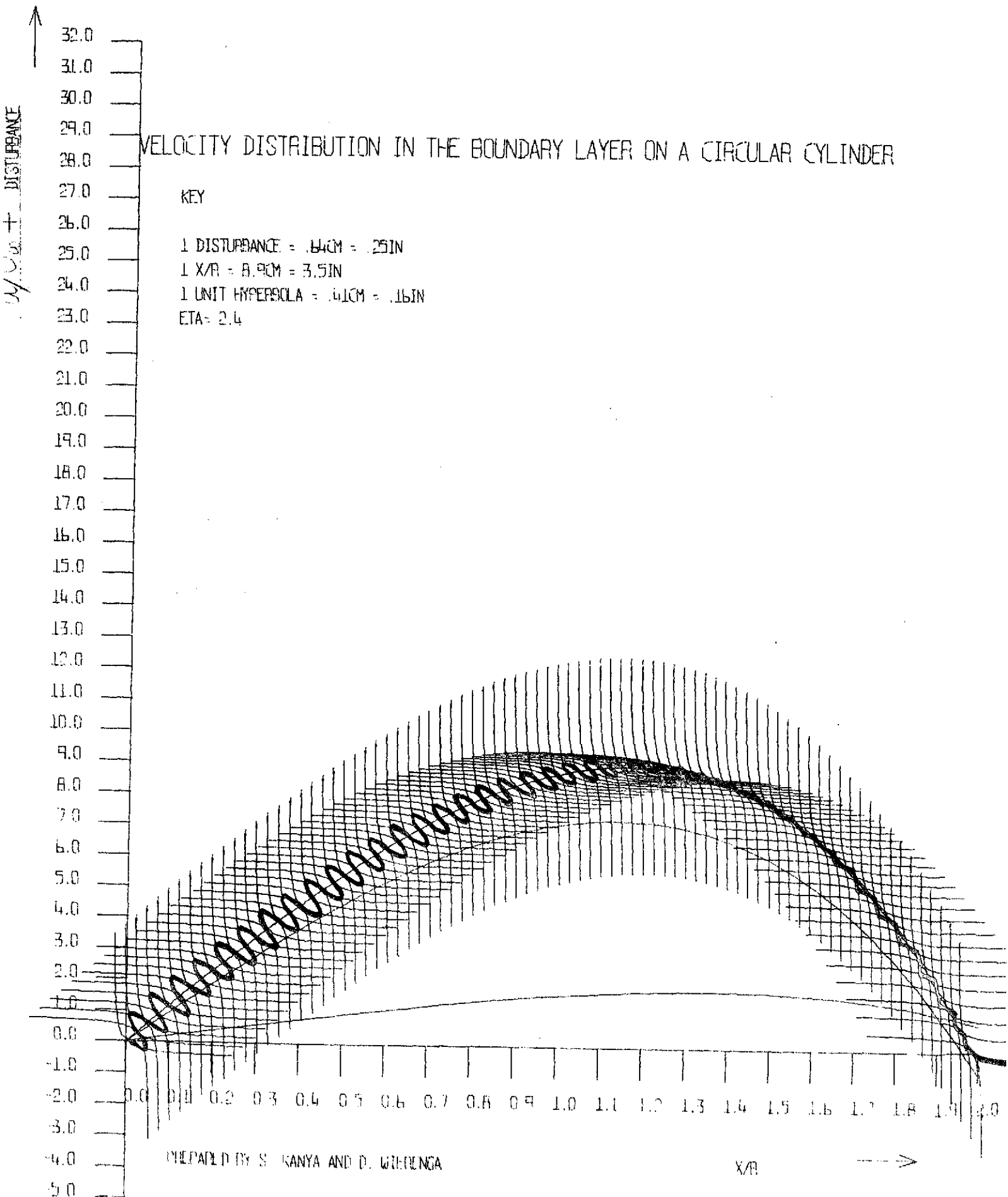
1 UNIT HYPERBOLA = $.41CM = .16IN$

ETA = 2.2



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X/R



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

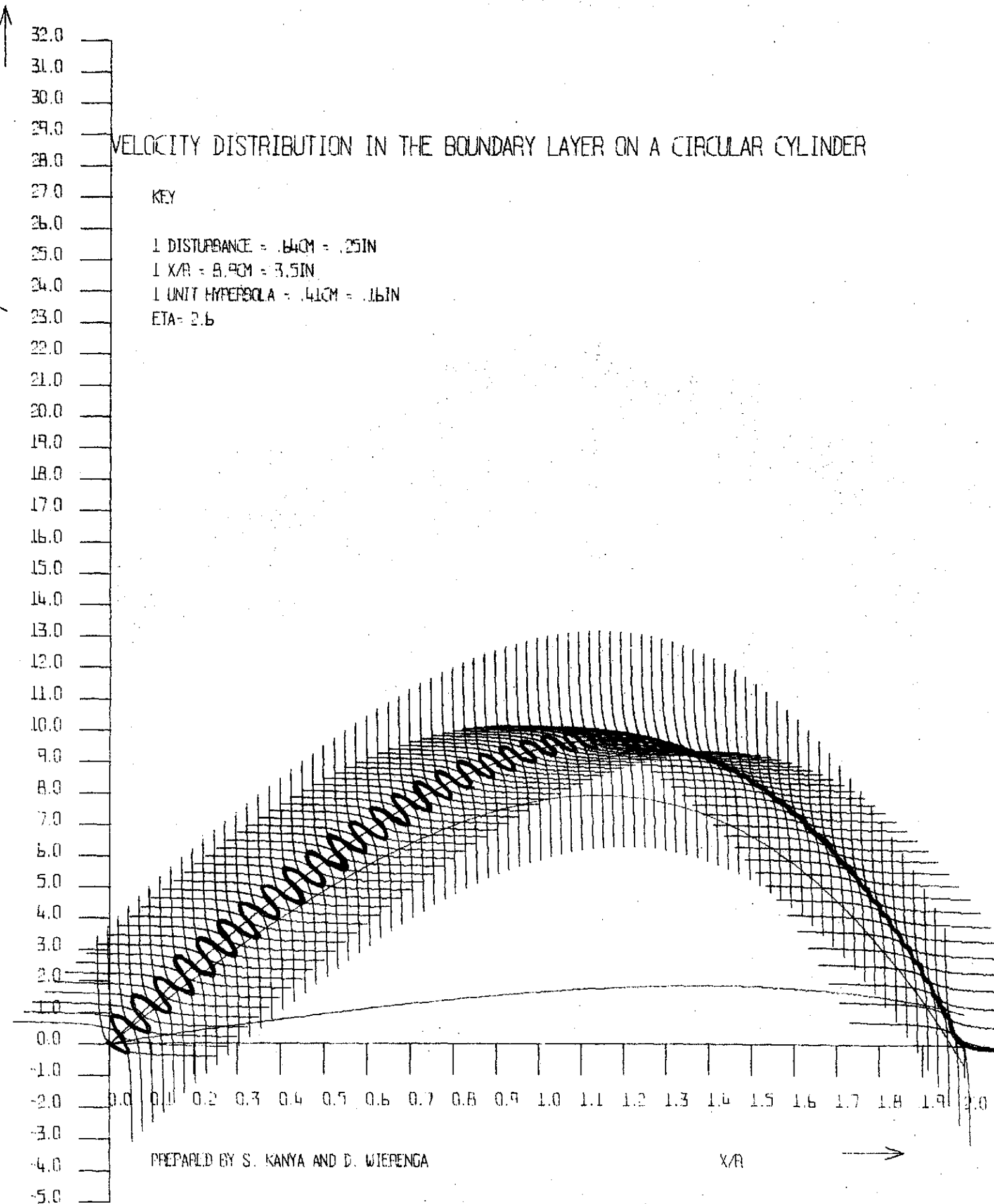
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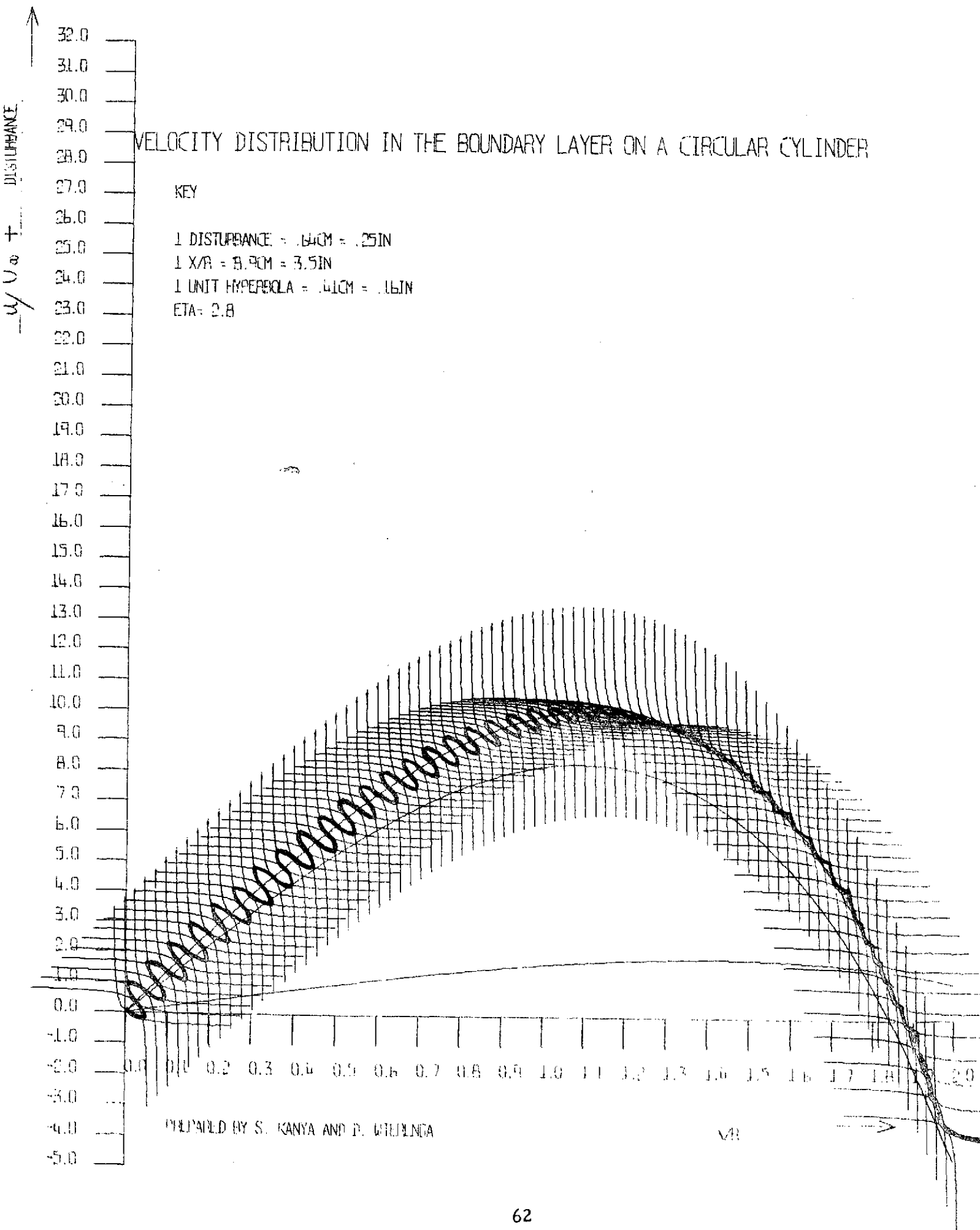
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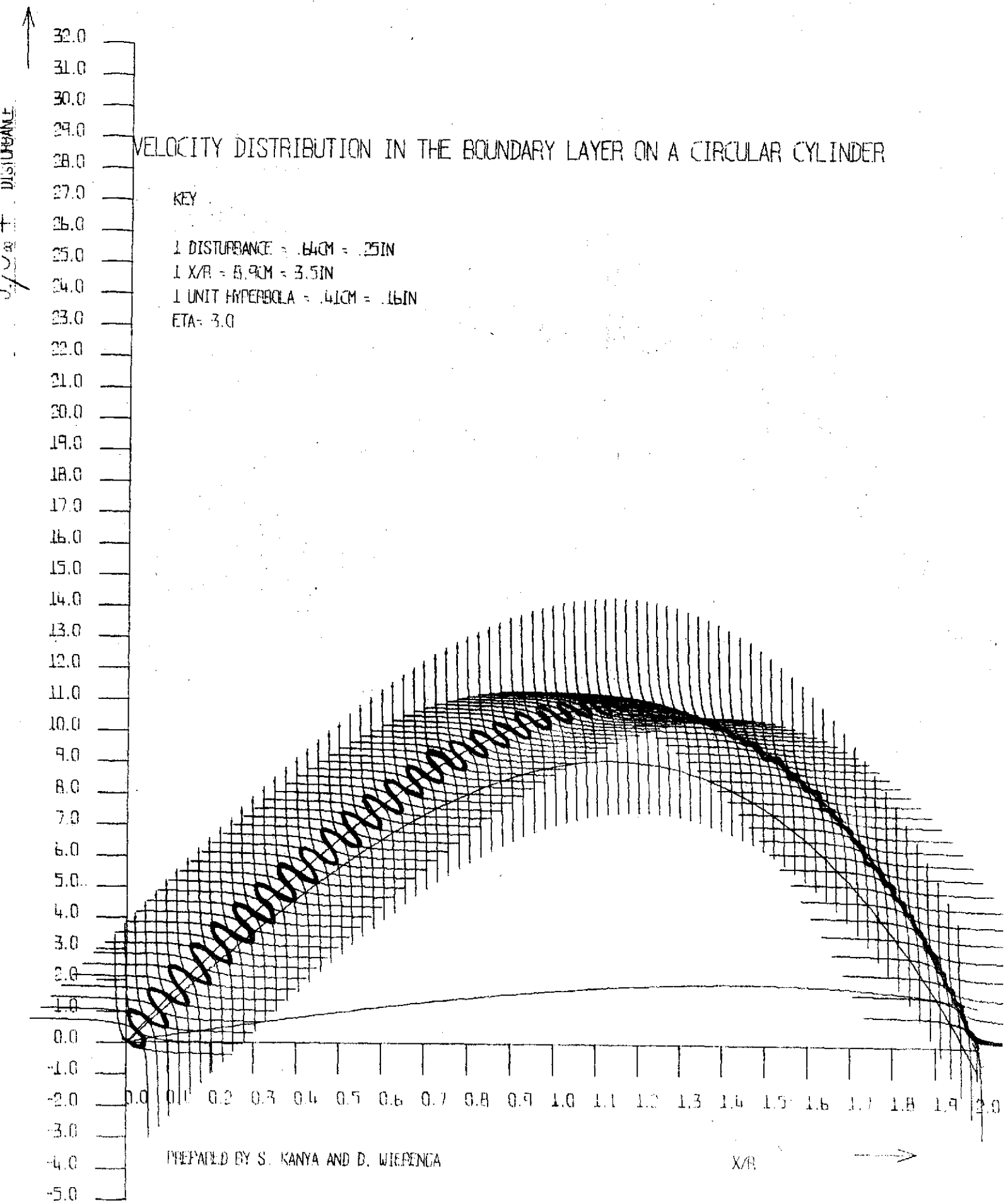
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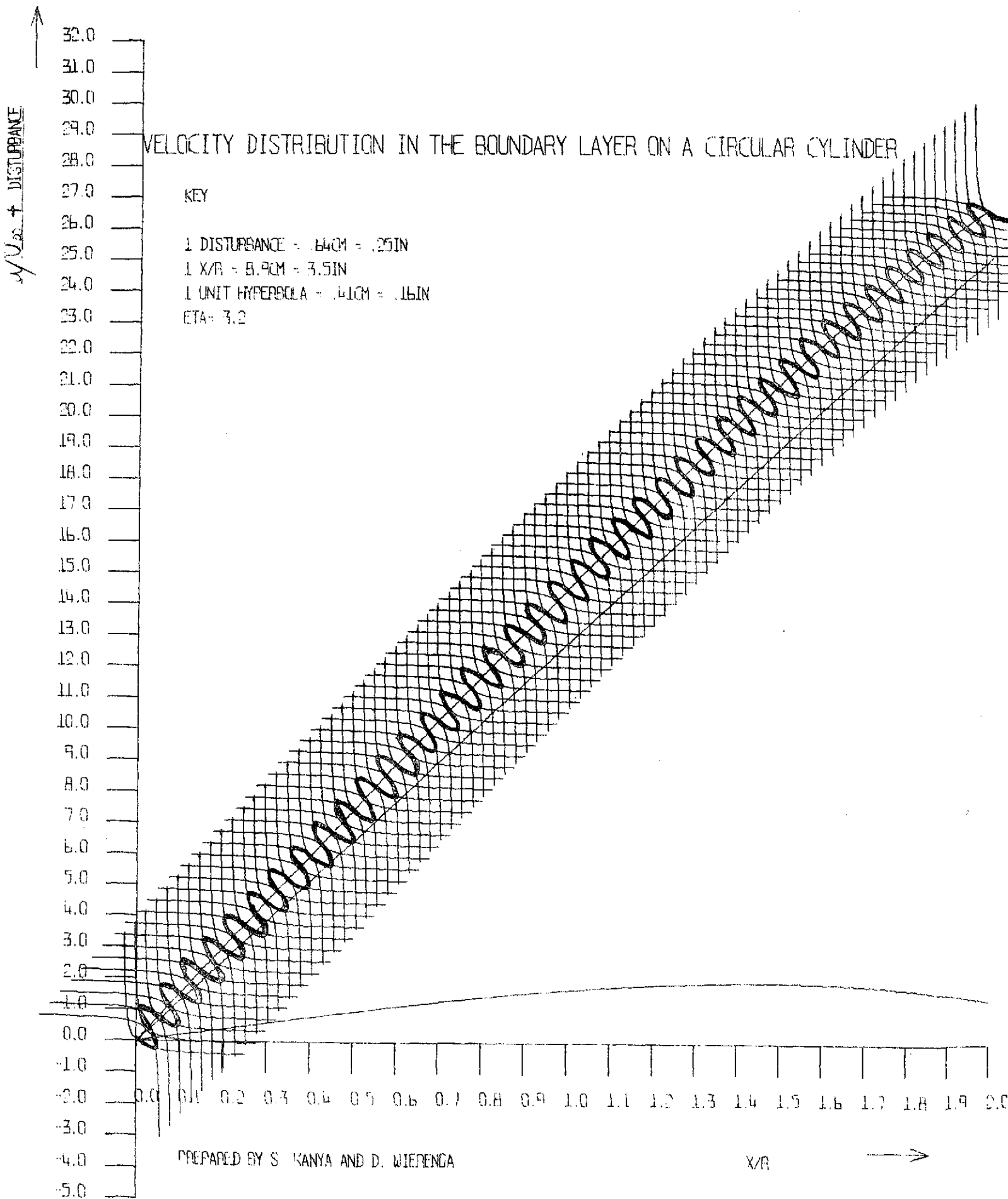
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ETA = 2.6







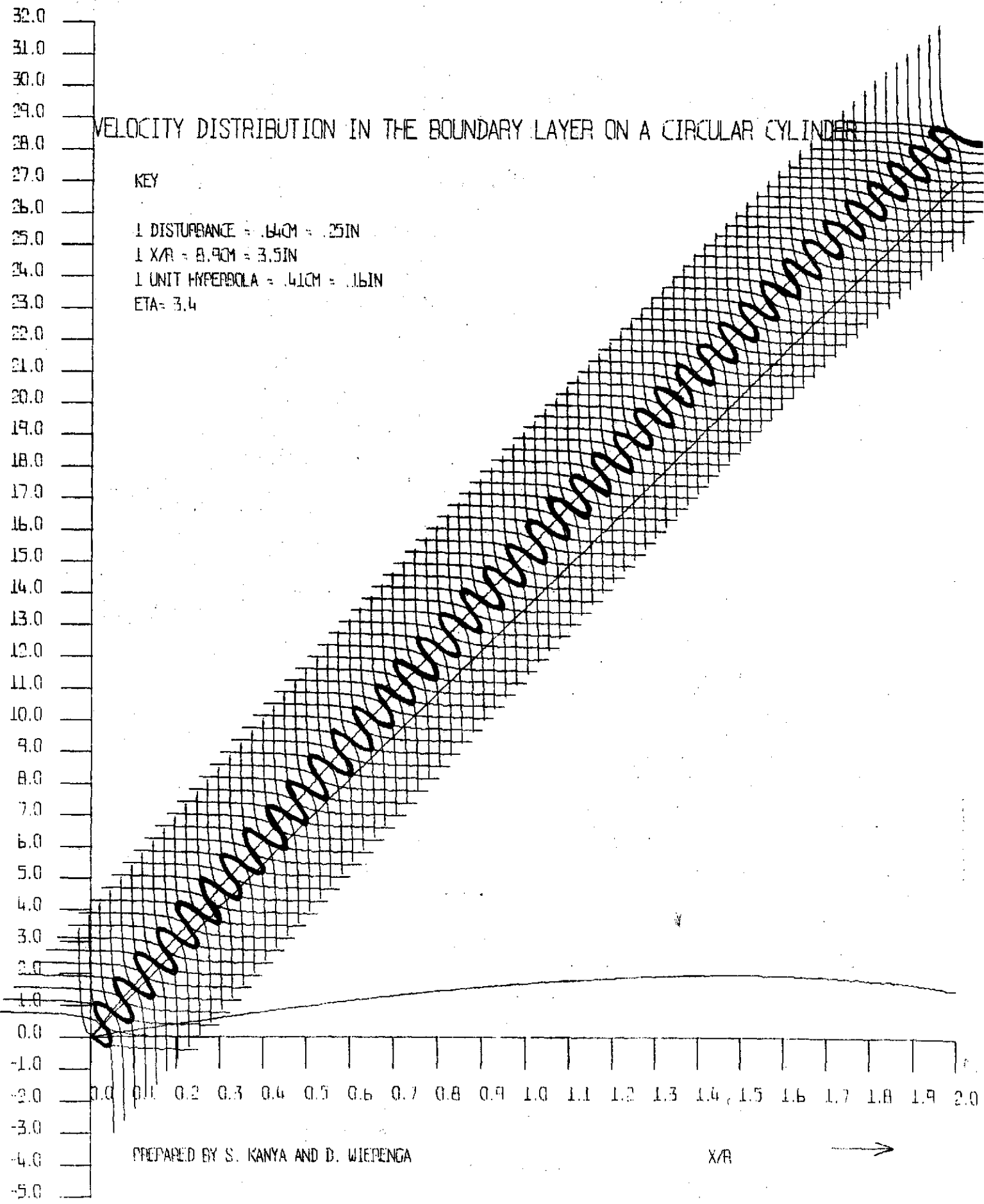


W. C. W. DIST. 100-100

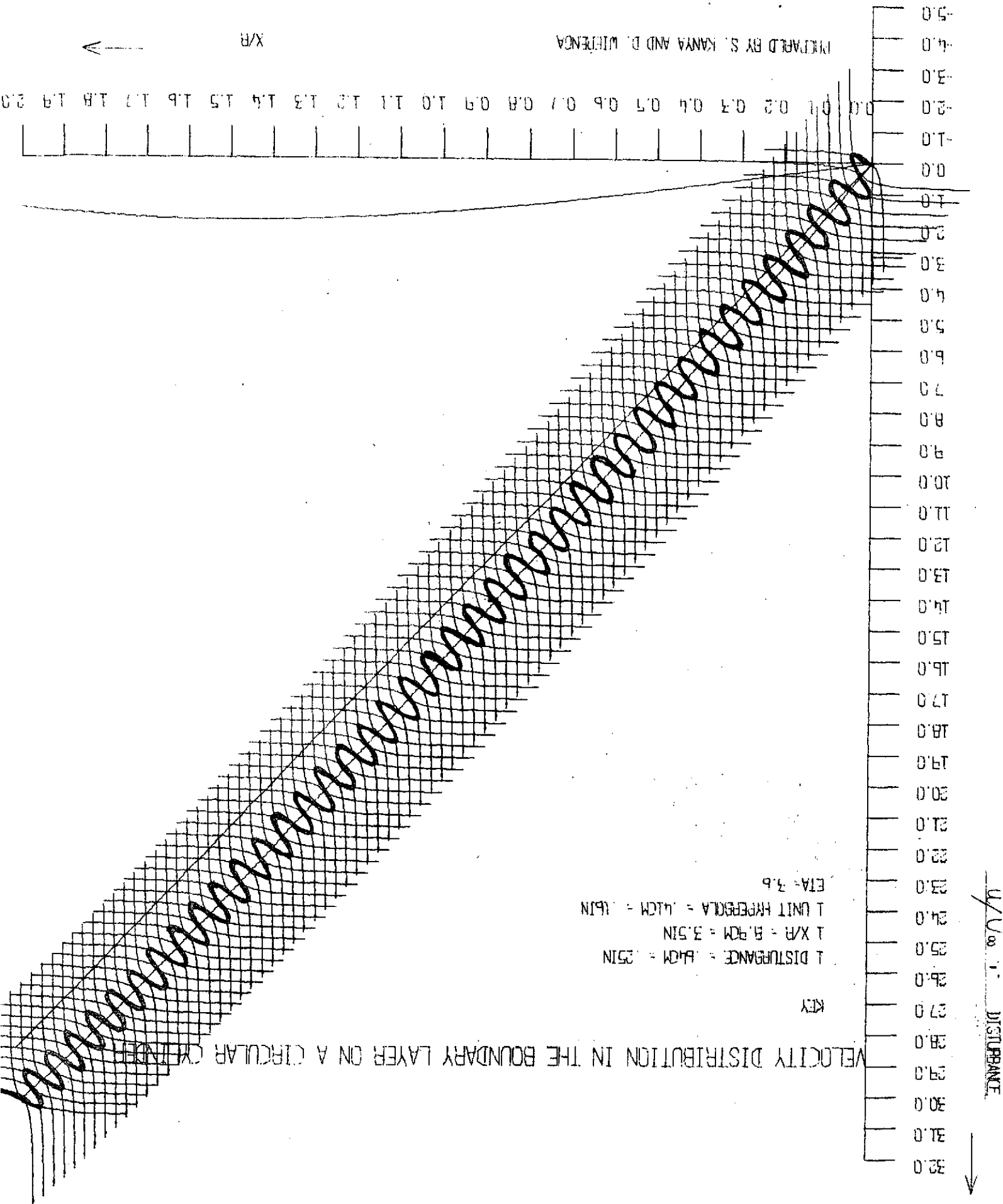
VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

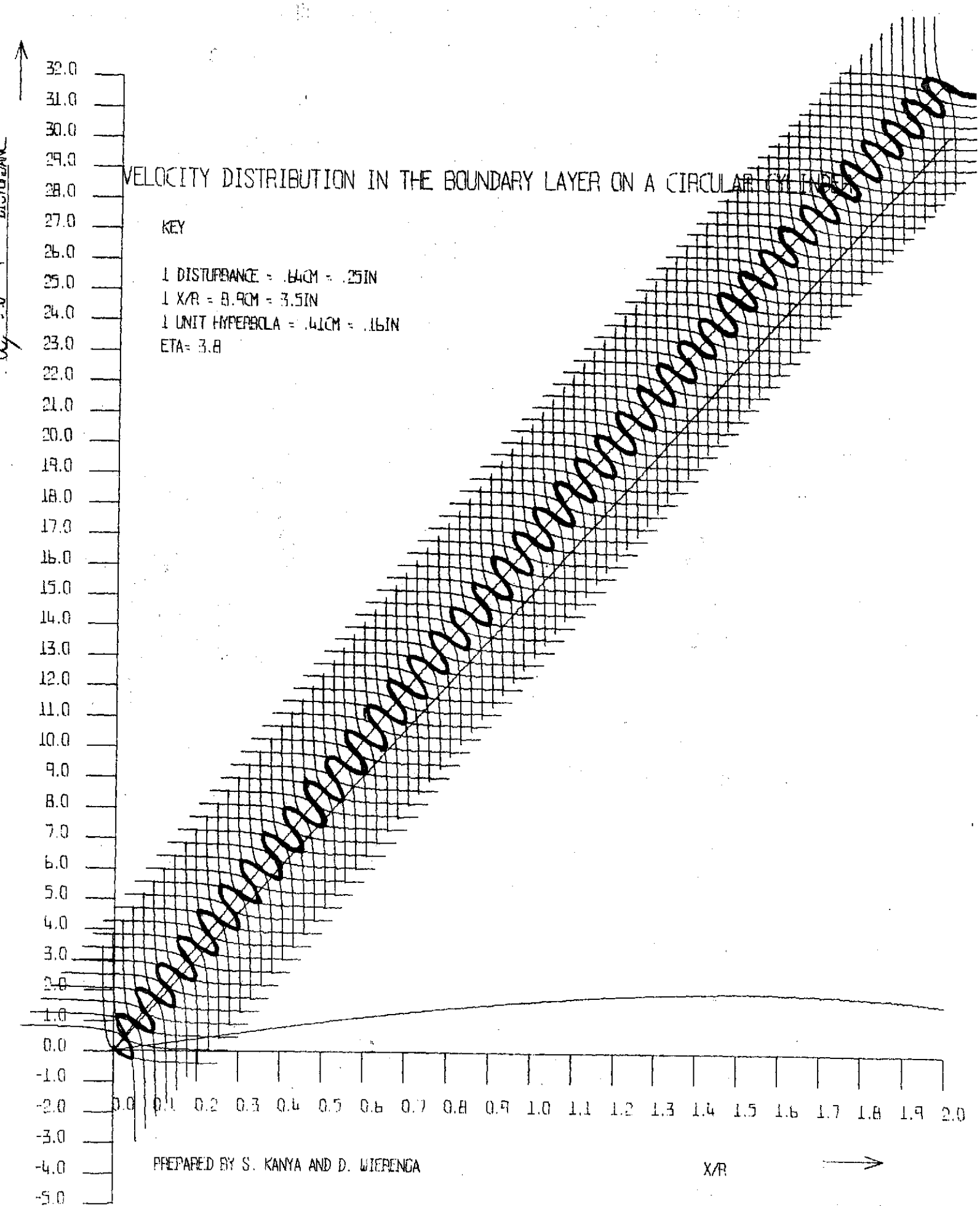
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- 1 X/R = 8.9CM = 3.5IN
- 1 UNIT HYPERBOLA = .41CM = .16IN
- ETA = 3.4

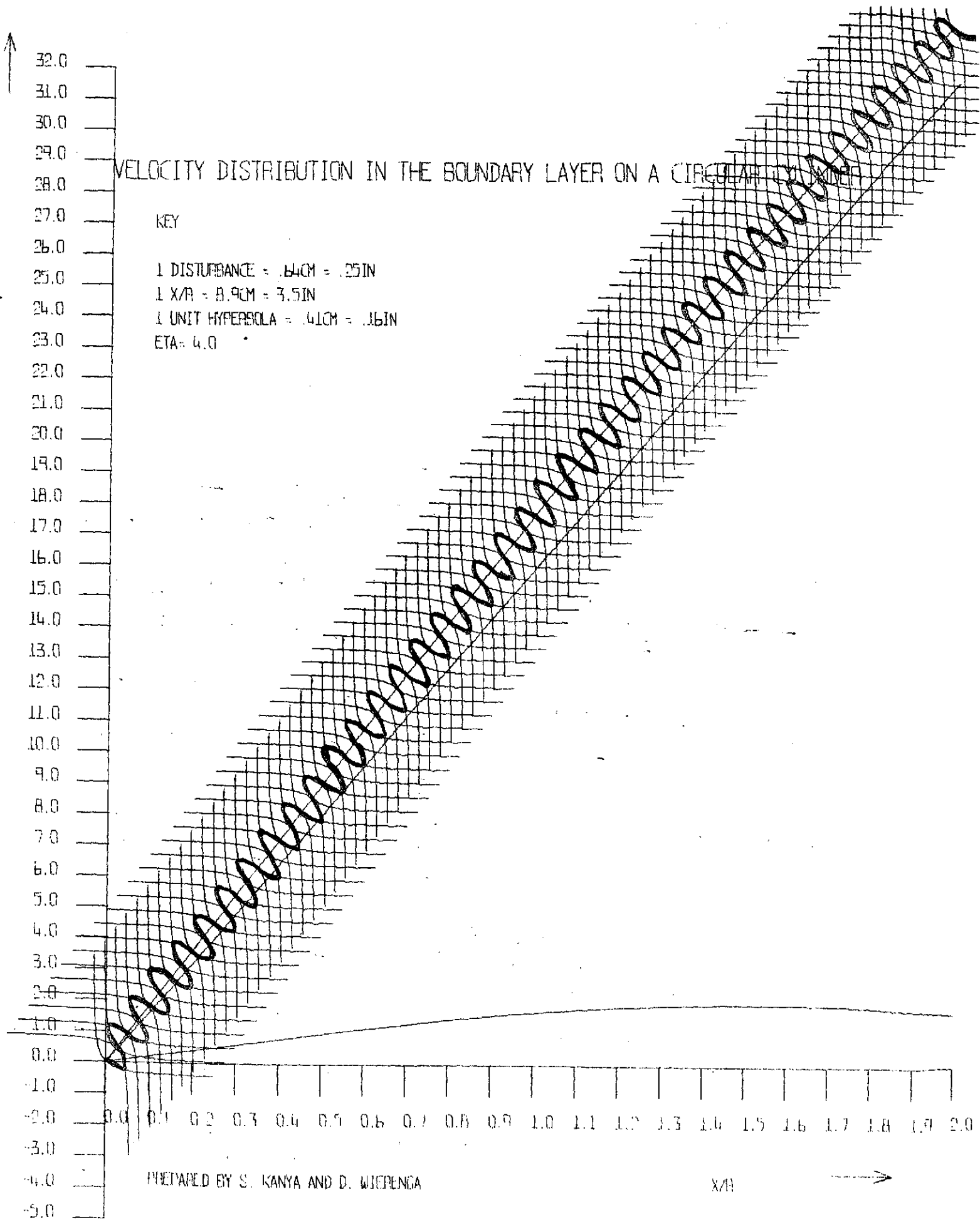


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WAVE DISTURBANCE



The Third Set of Plots

The function ($v_d + \text{disturbance}$), $v_d = v U_\infty^{-1}$, as the function of ($x R^{-1}$) for various values of the parameter $\eta = 0.2, 0.4, \dots, 4.0$. The function $v_d = v U_\infty^{-1}$ is dimensionless, the disturbance, equal to ω_z , is superimposed upon the dimensionless velocity function, $v_d = v U_\infty^{-1}$. Time is frozen.

VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

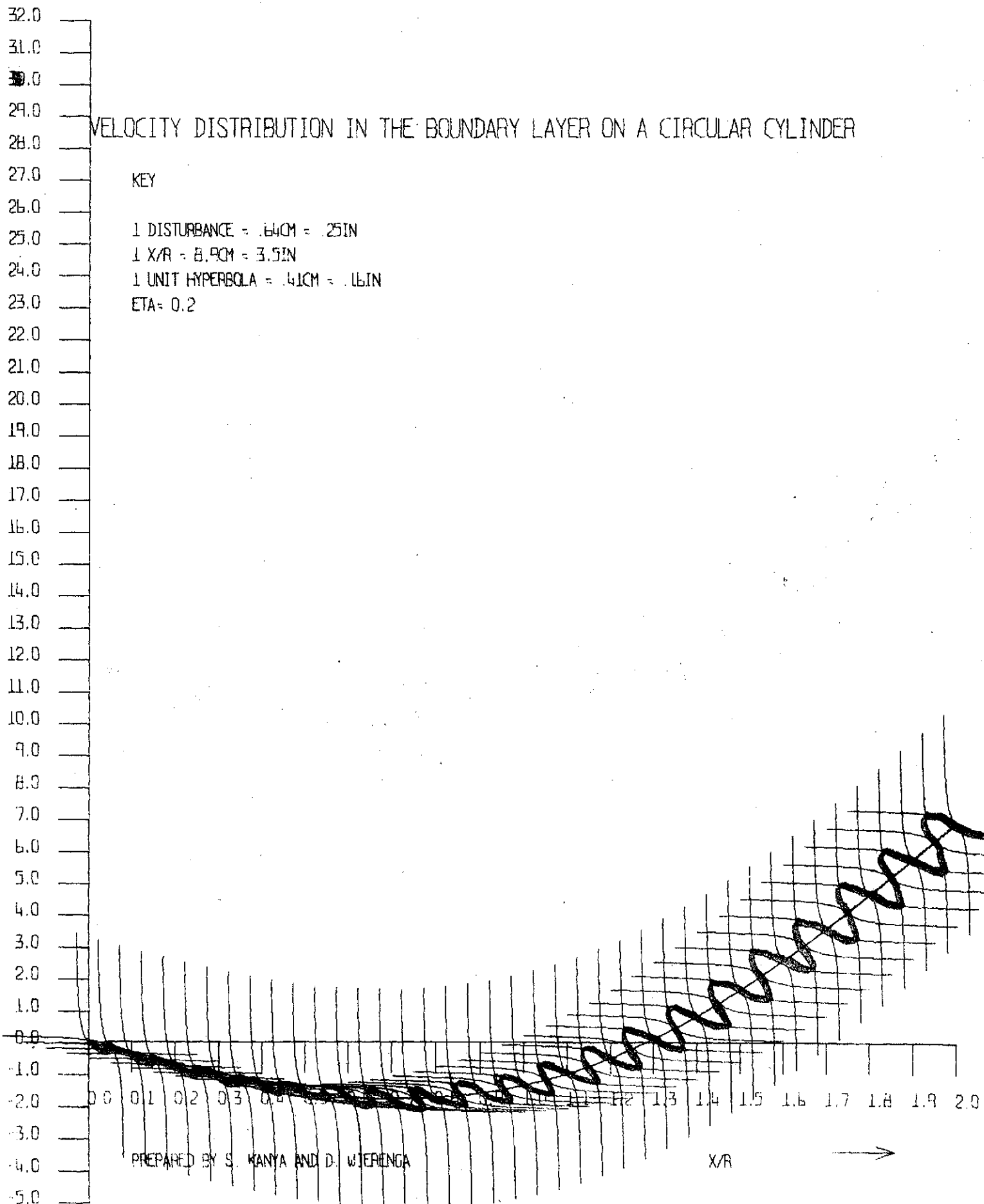
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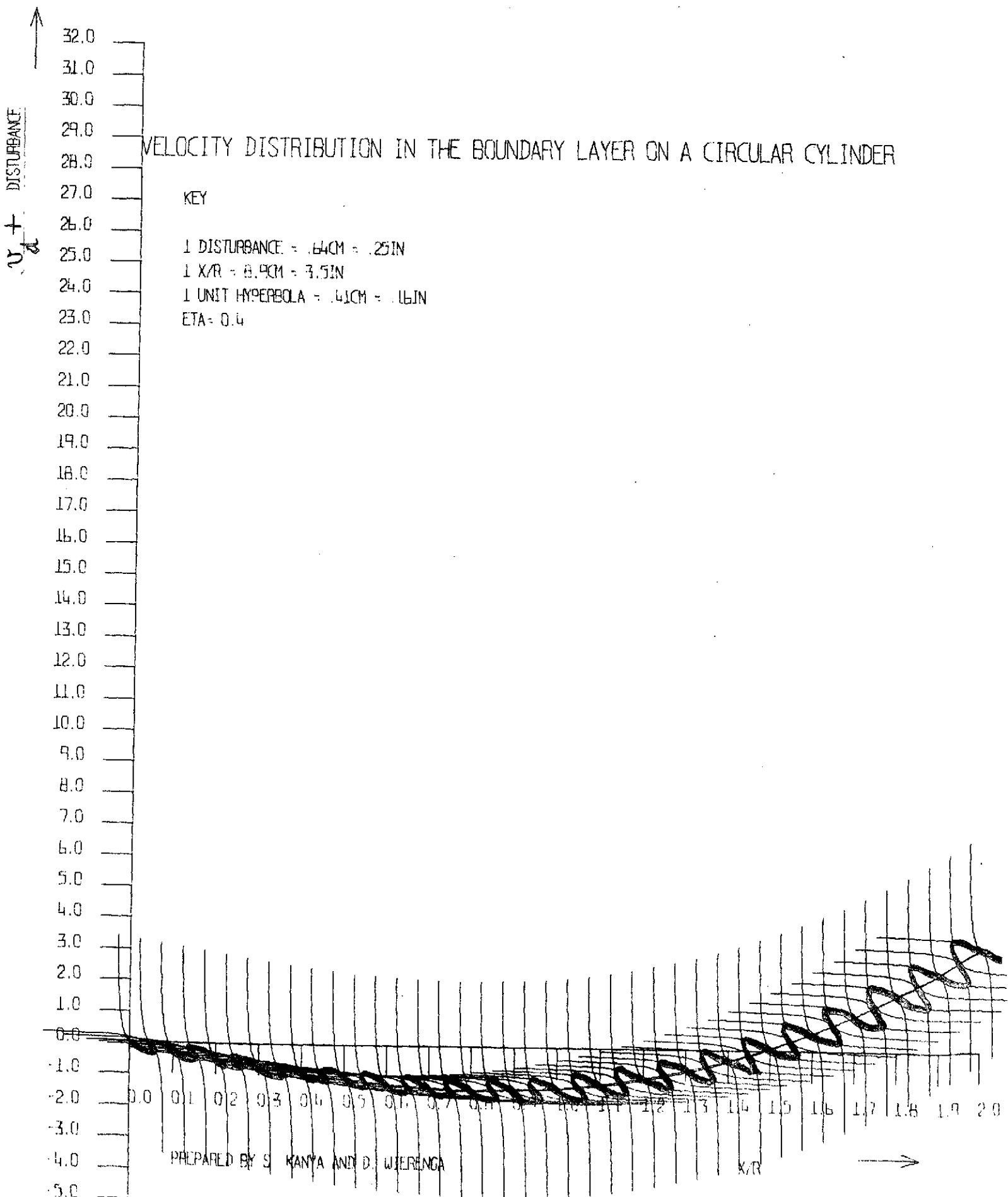
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1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 0.2





$U_d + \text{DISTURBANCE}$

32.0
31.0
30.0
29.0
28.0
27.0
26.0
25.0
24.0
23.0
22.0
21.0
20.0
19.0
18.0
17.0
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
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3.0
2.0
1.0
0.0
-1.0
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-3.0
-4.0
-5.0

VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

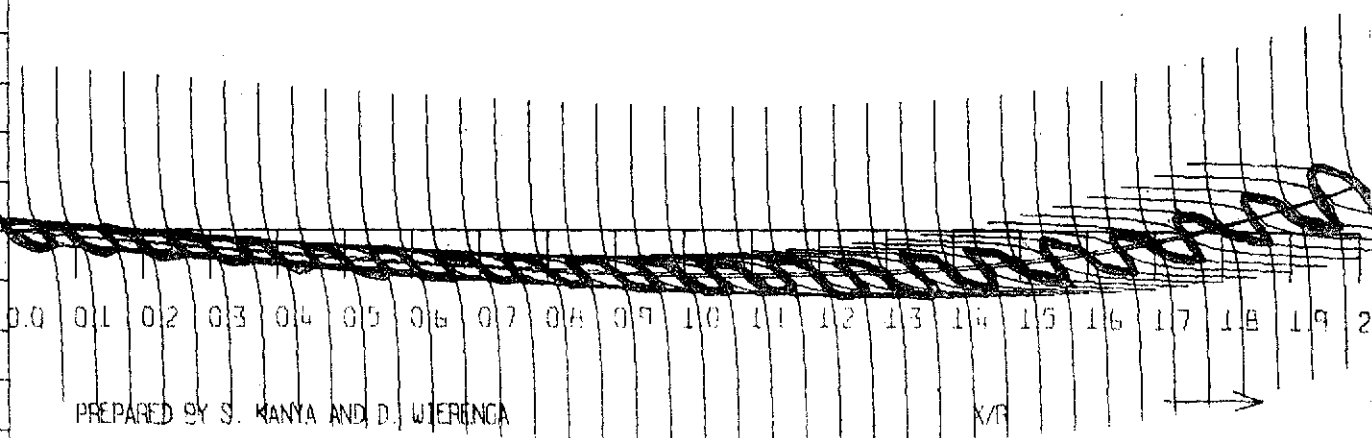
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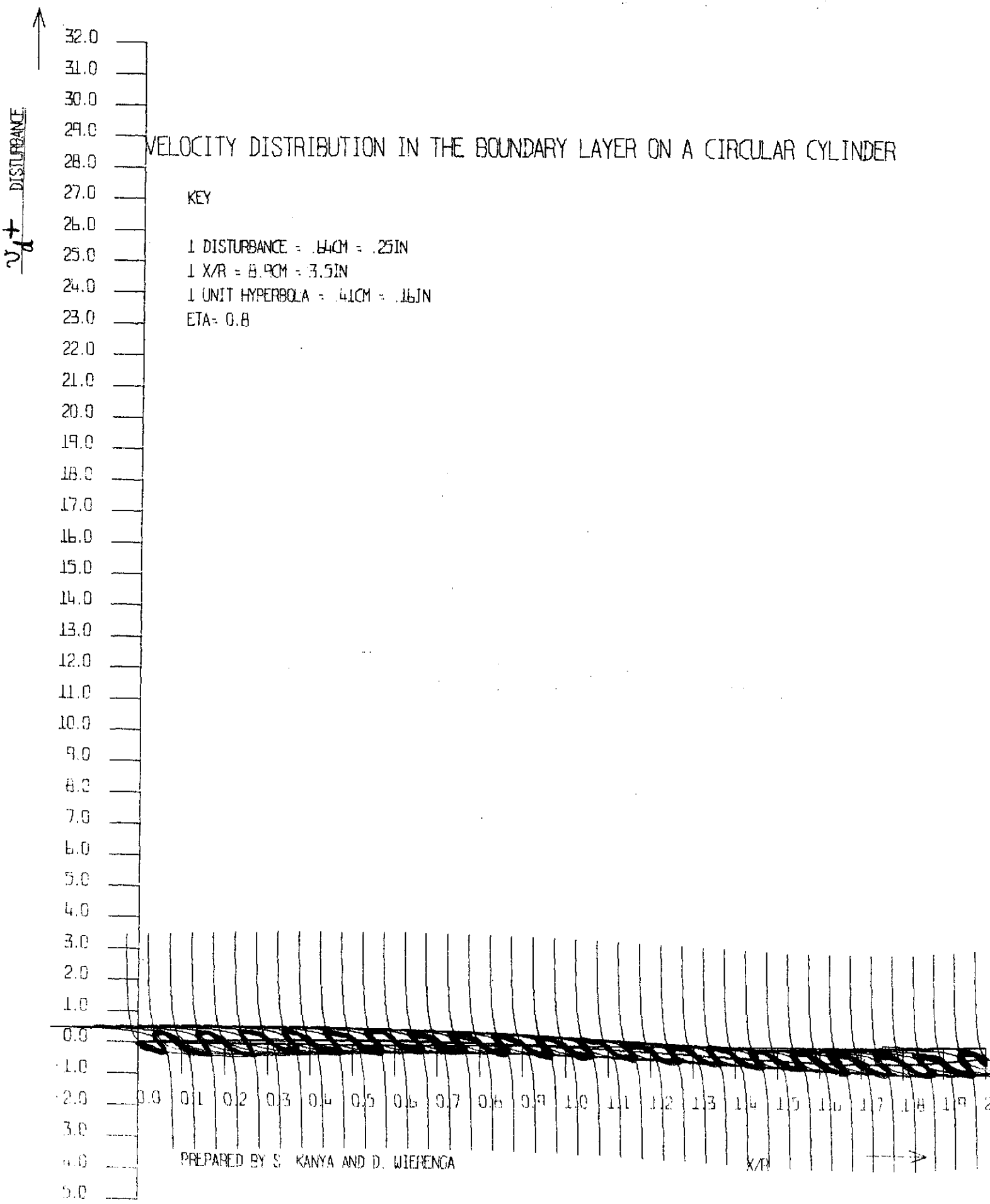
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1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 0.6





VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

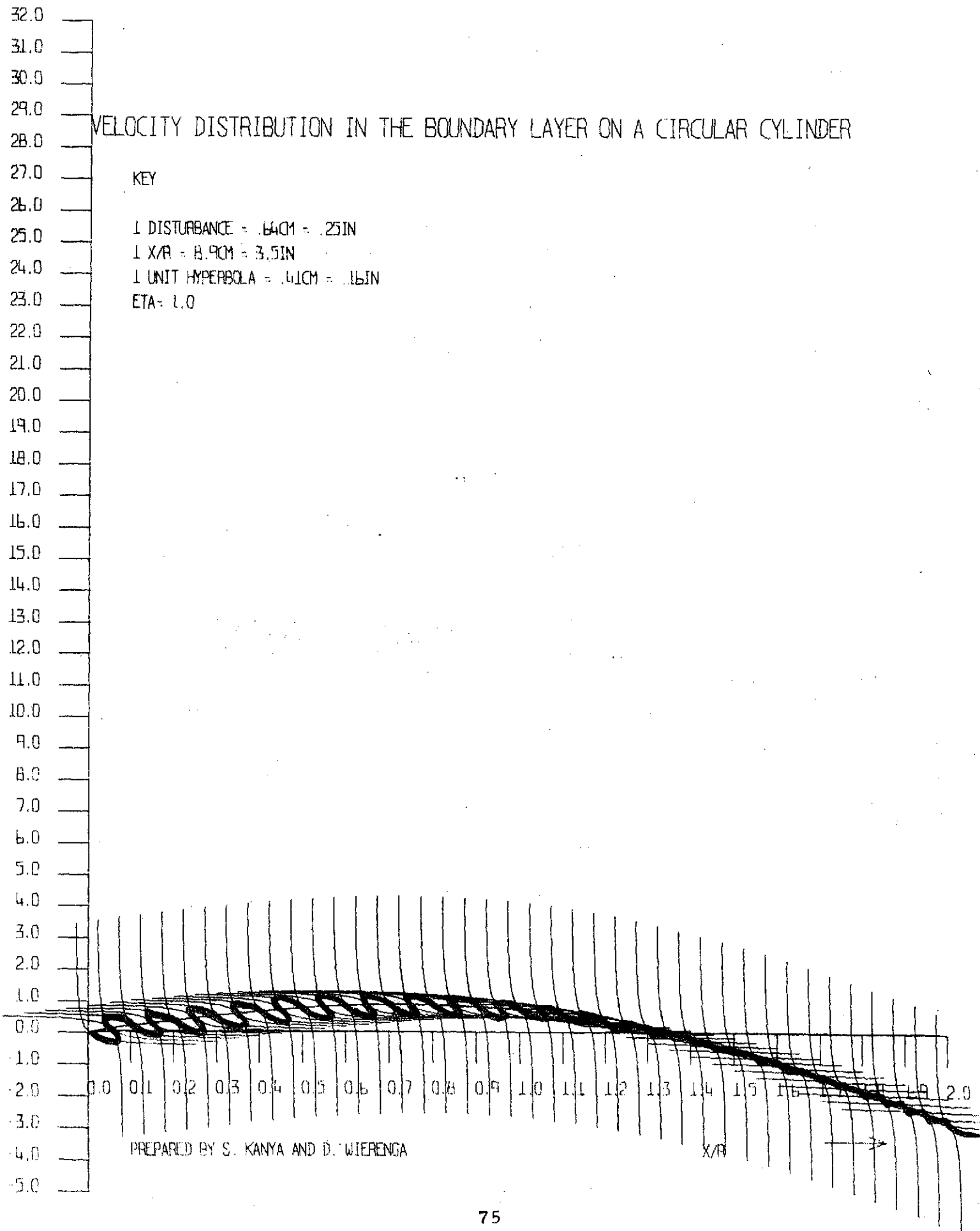
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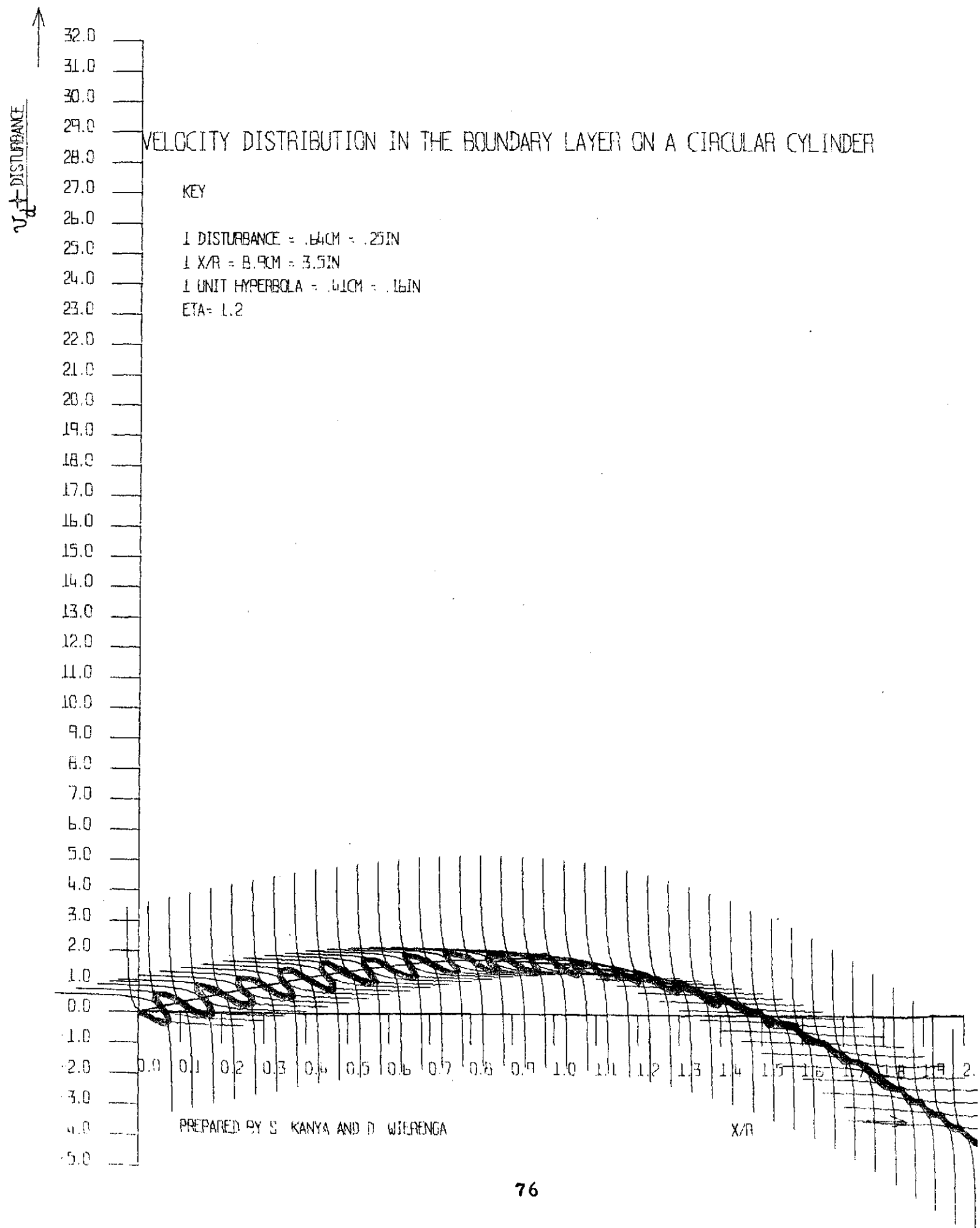
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1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 1.0





VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

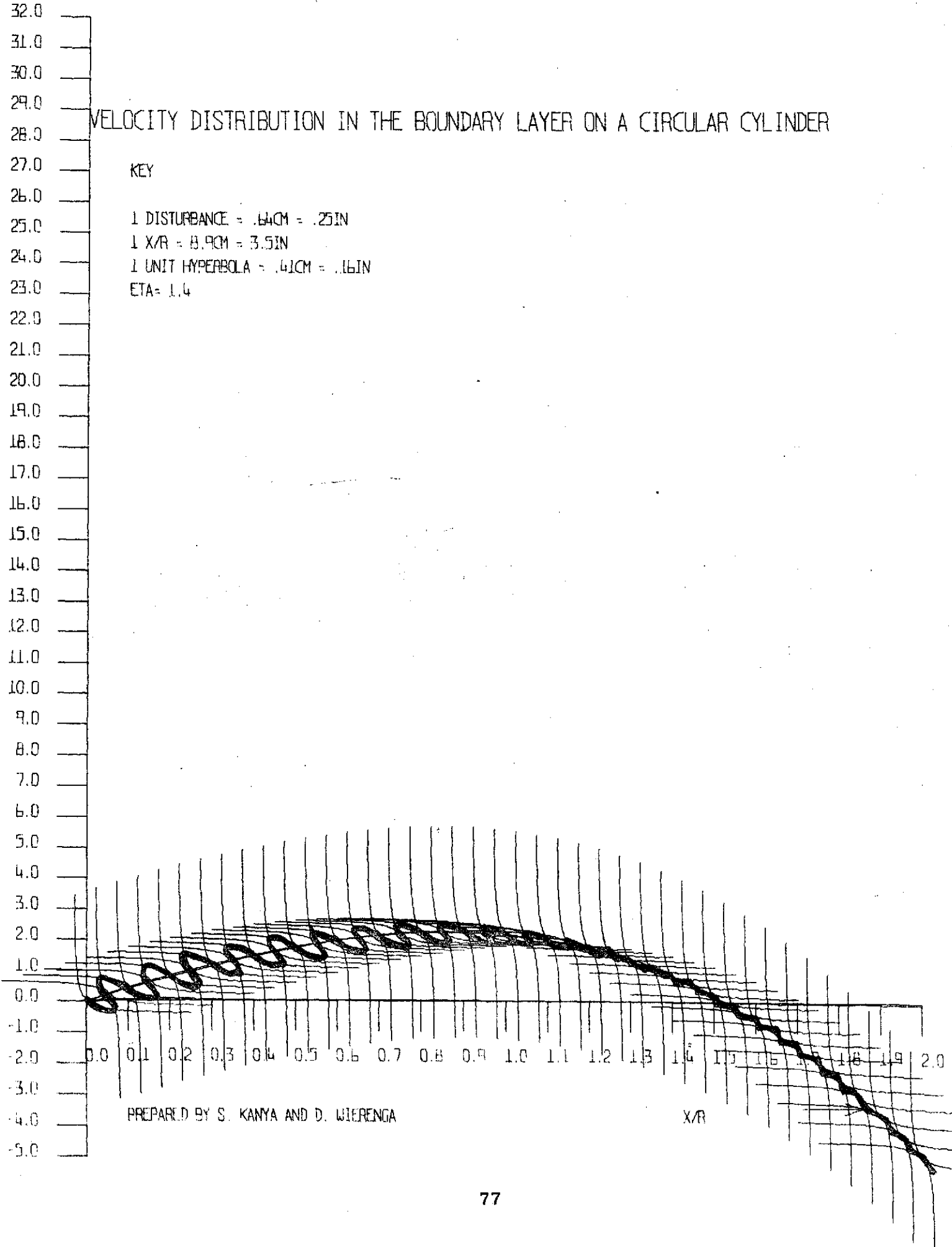
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1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .64CM = .25IN

ETA = 1.4



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

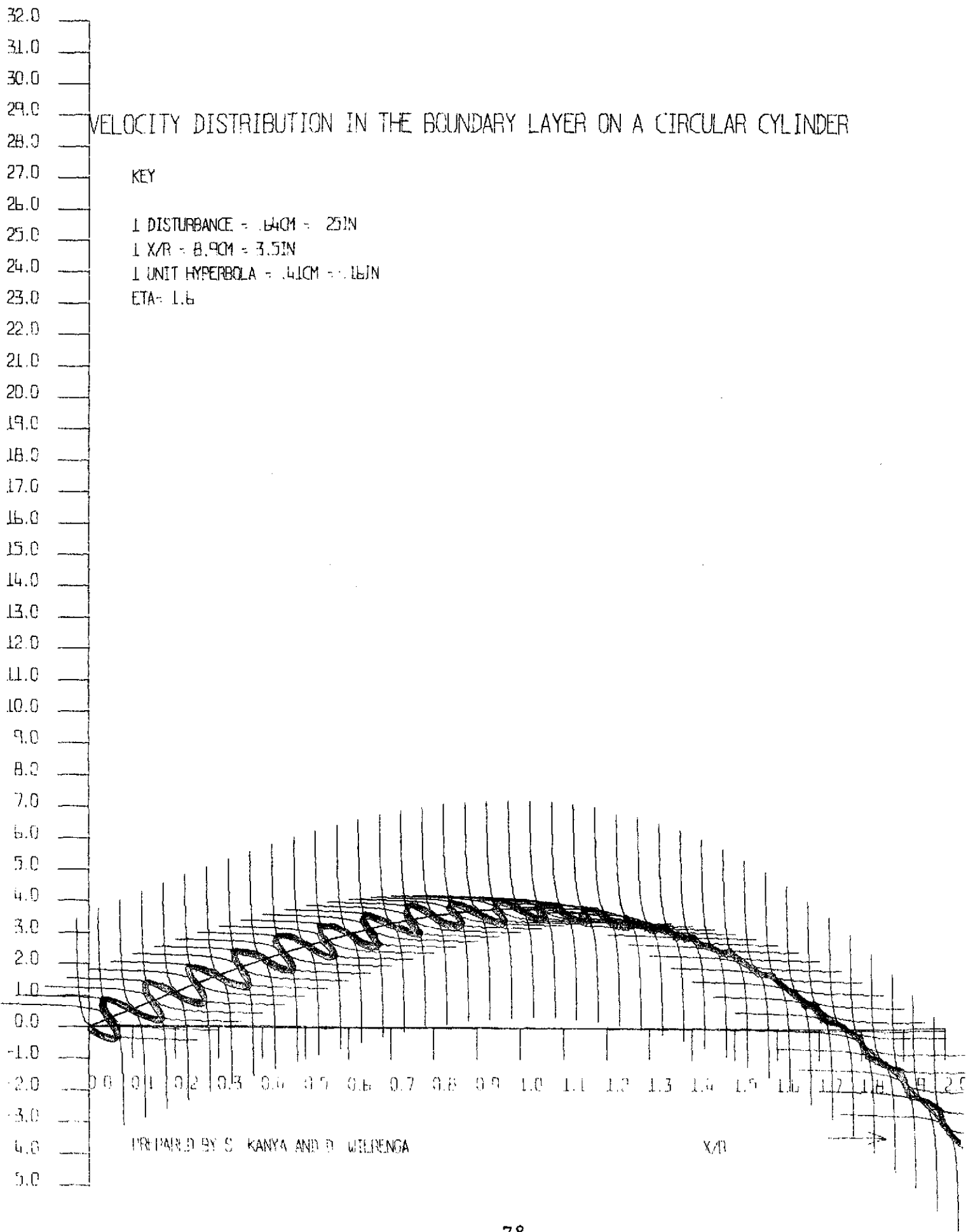
KEY

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1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 1.6



$y + \text{DISTURBANCE}$

VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

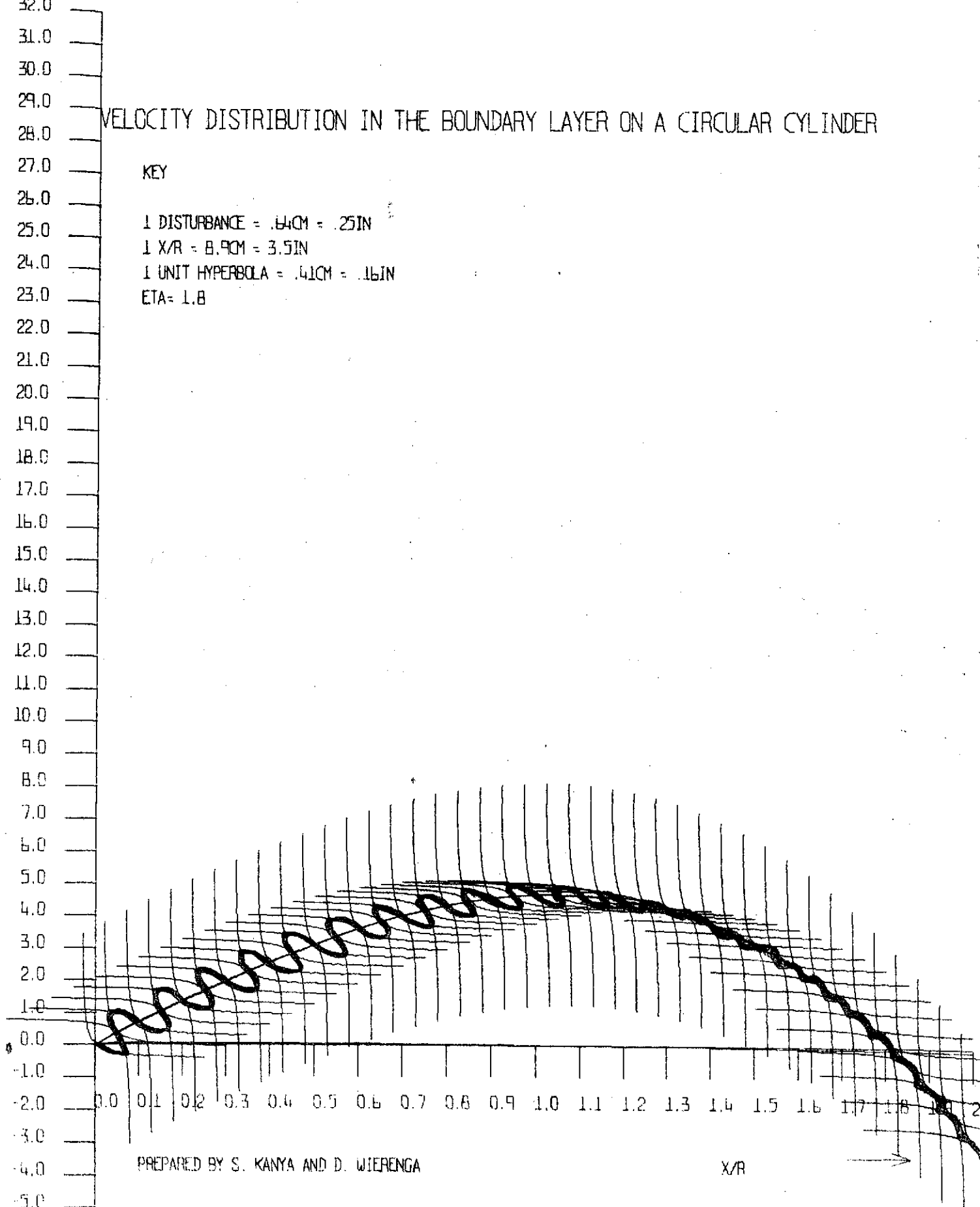
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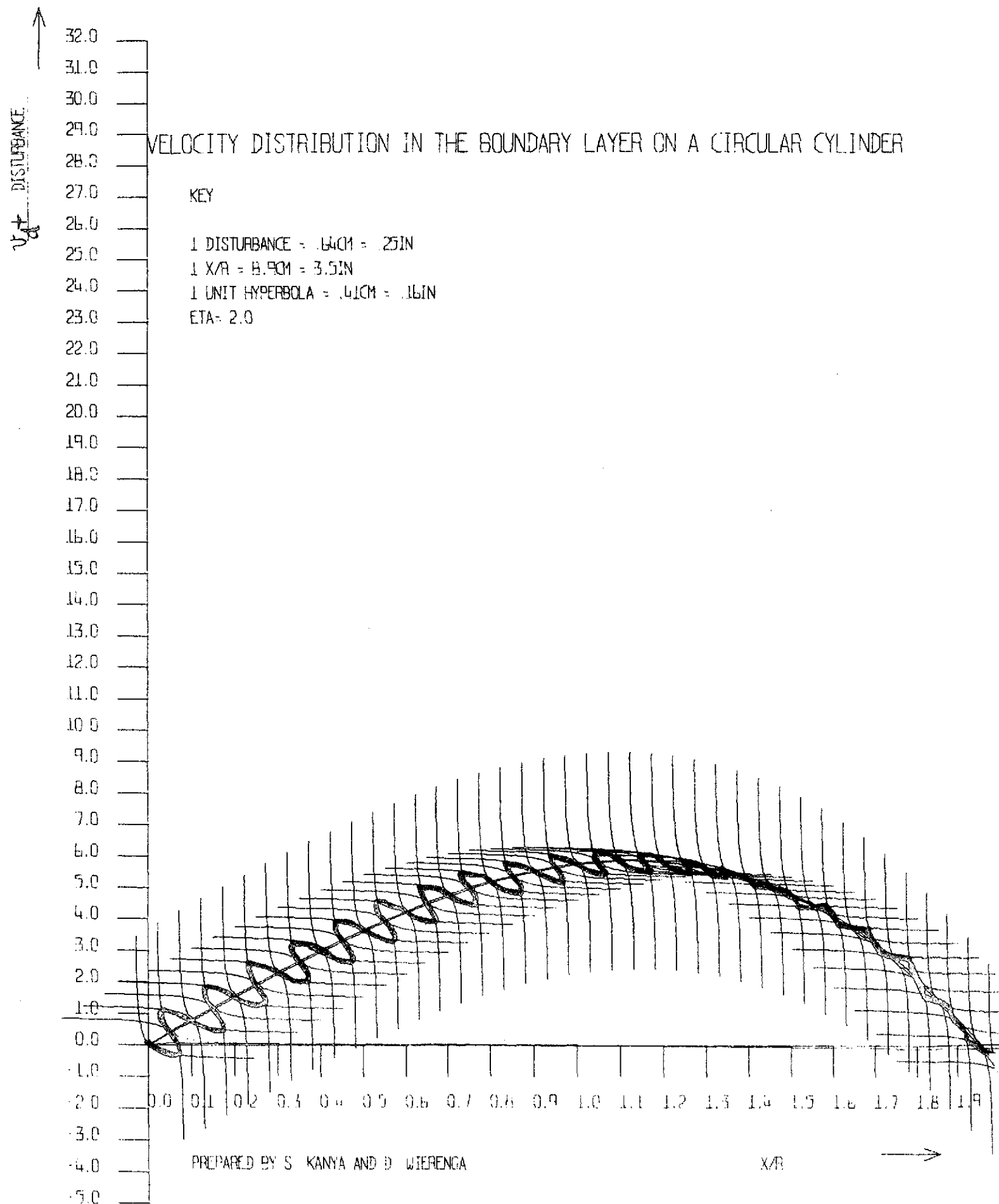
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1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 1.8





VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

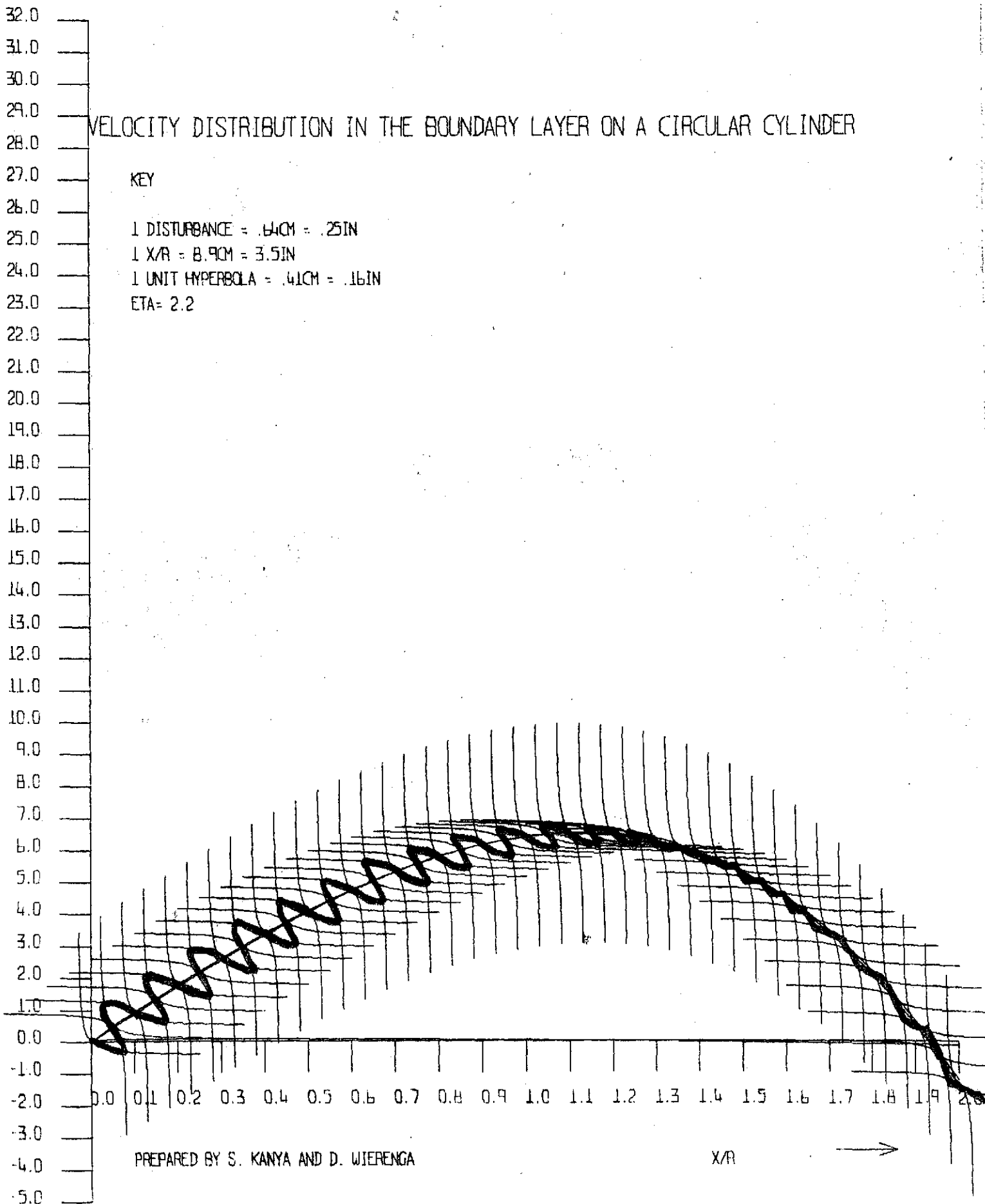
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1 X/R = 8.9CM = 3.5IN

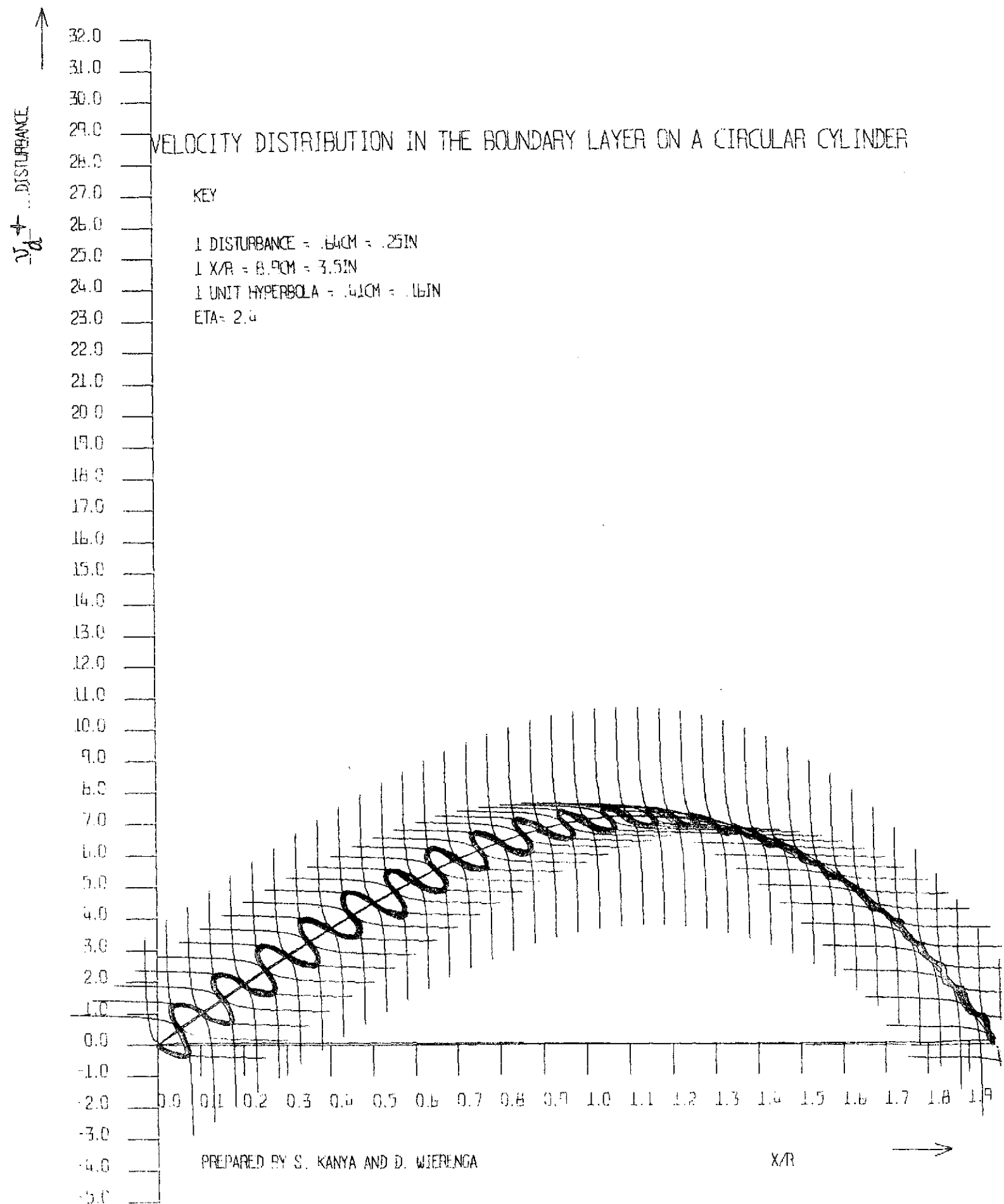
1 UNIT HYPERBOLA = .41CM = .16IN

ETA= 2.2



PREPARED BY S. KANYA AND D. WIERENGA

X/R



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

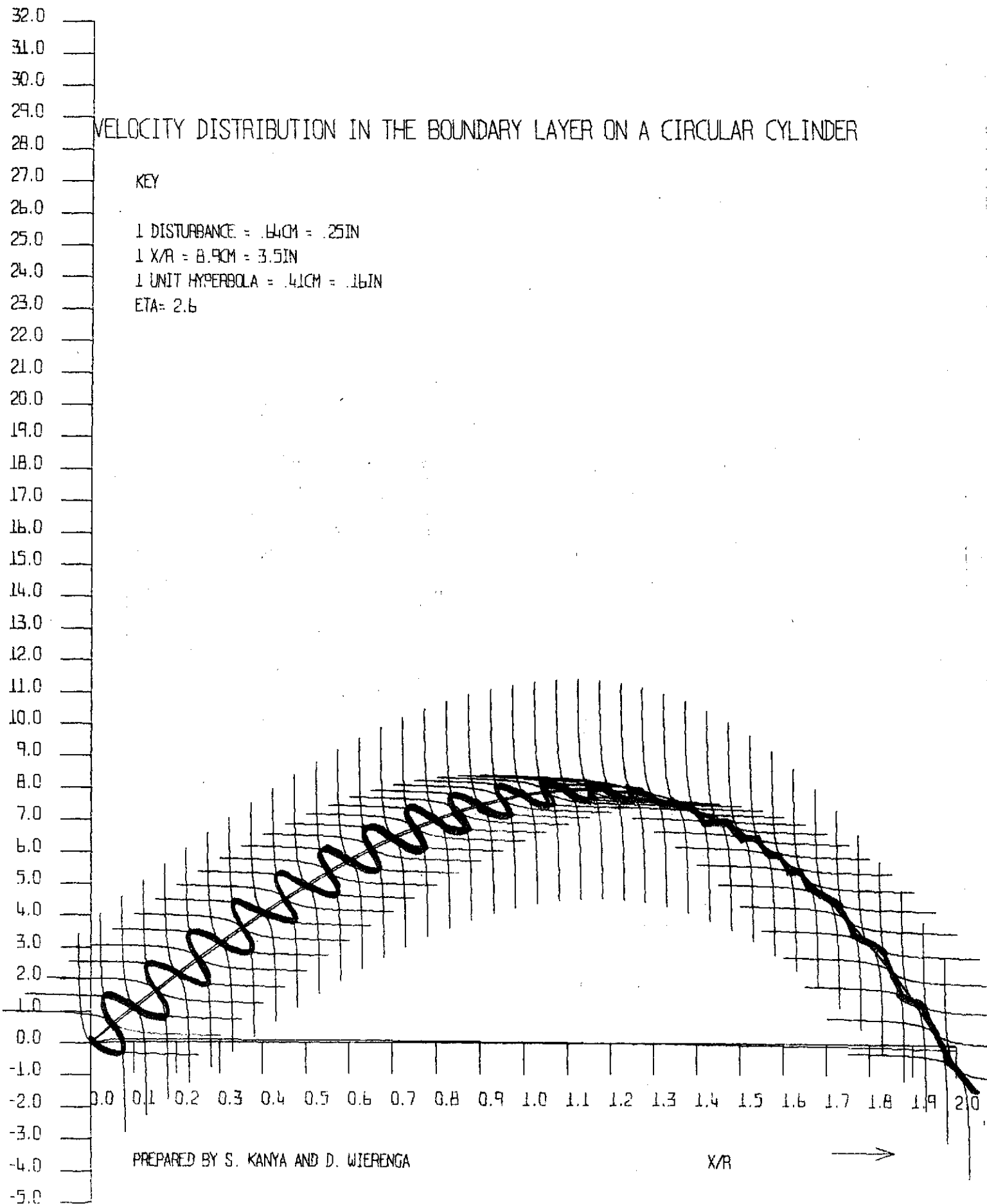
KEY

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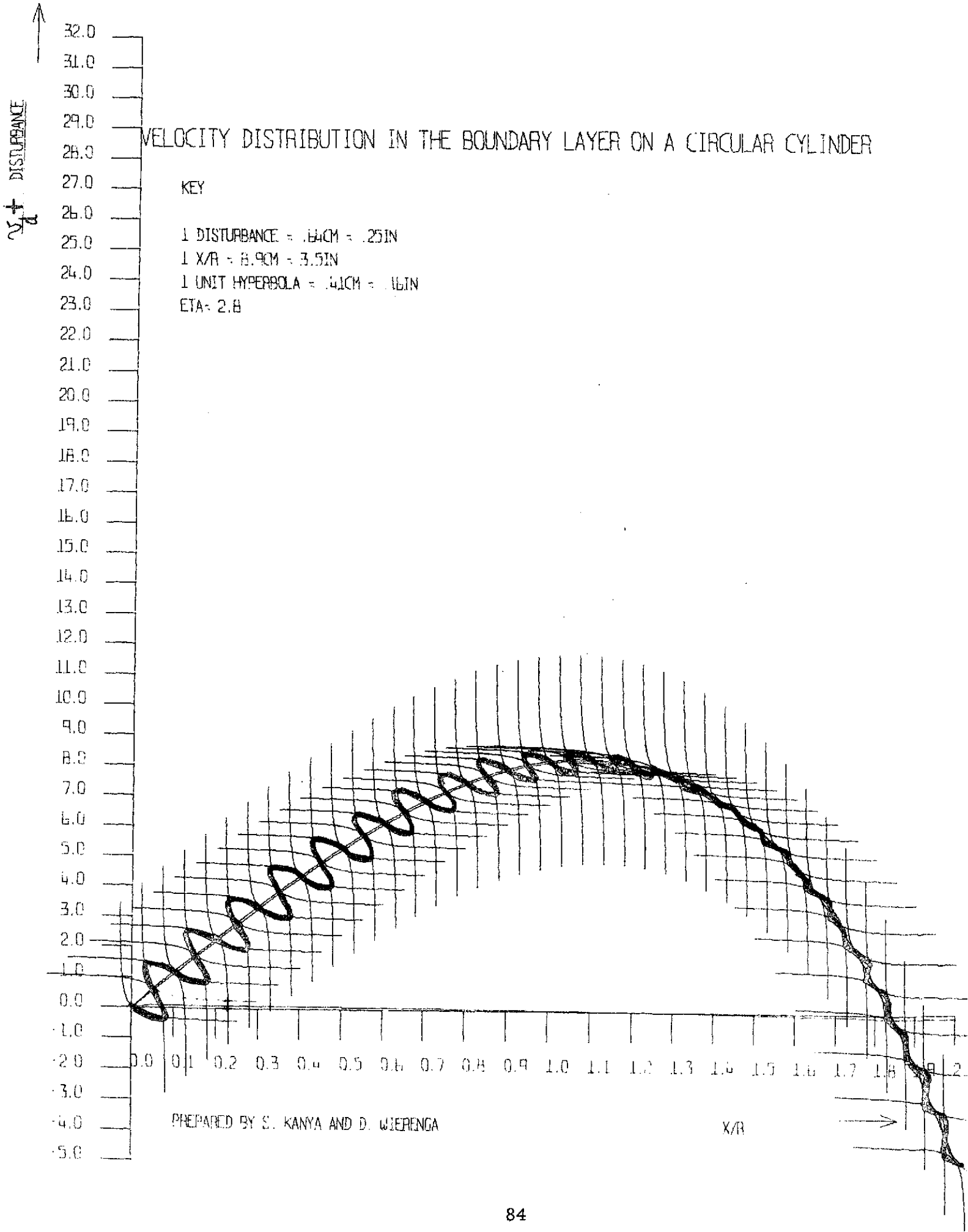
1 UNIT HYPERBOLA = .41CM = .16IN

ETA= 2.6



PREPARED BY S. KANYA AND D. WIERENGA

X/R



VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

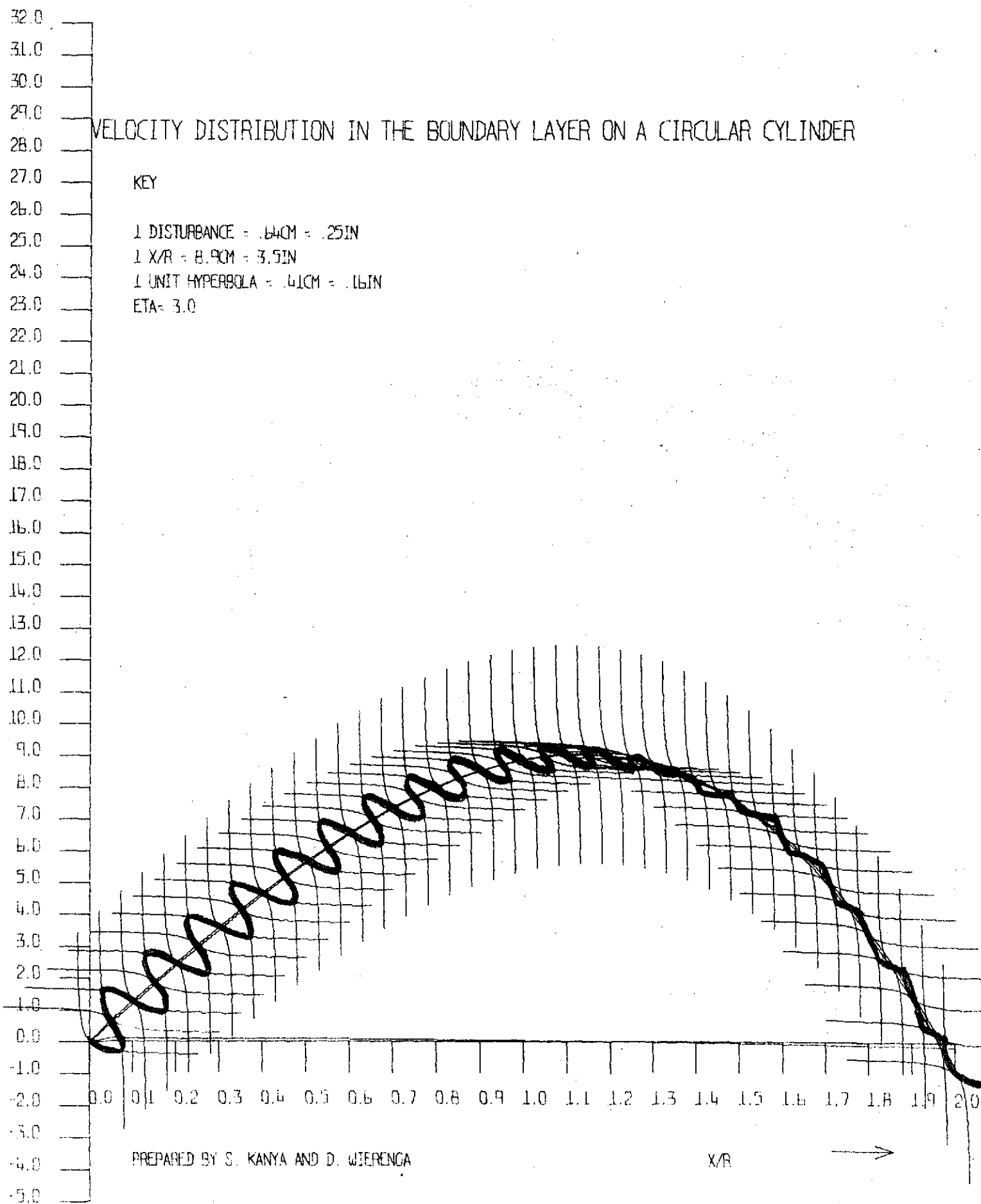
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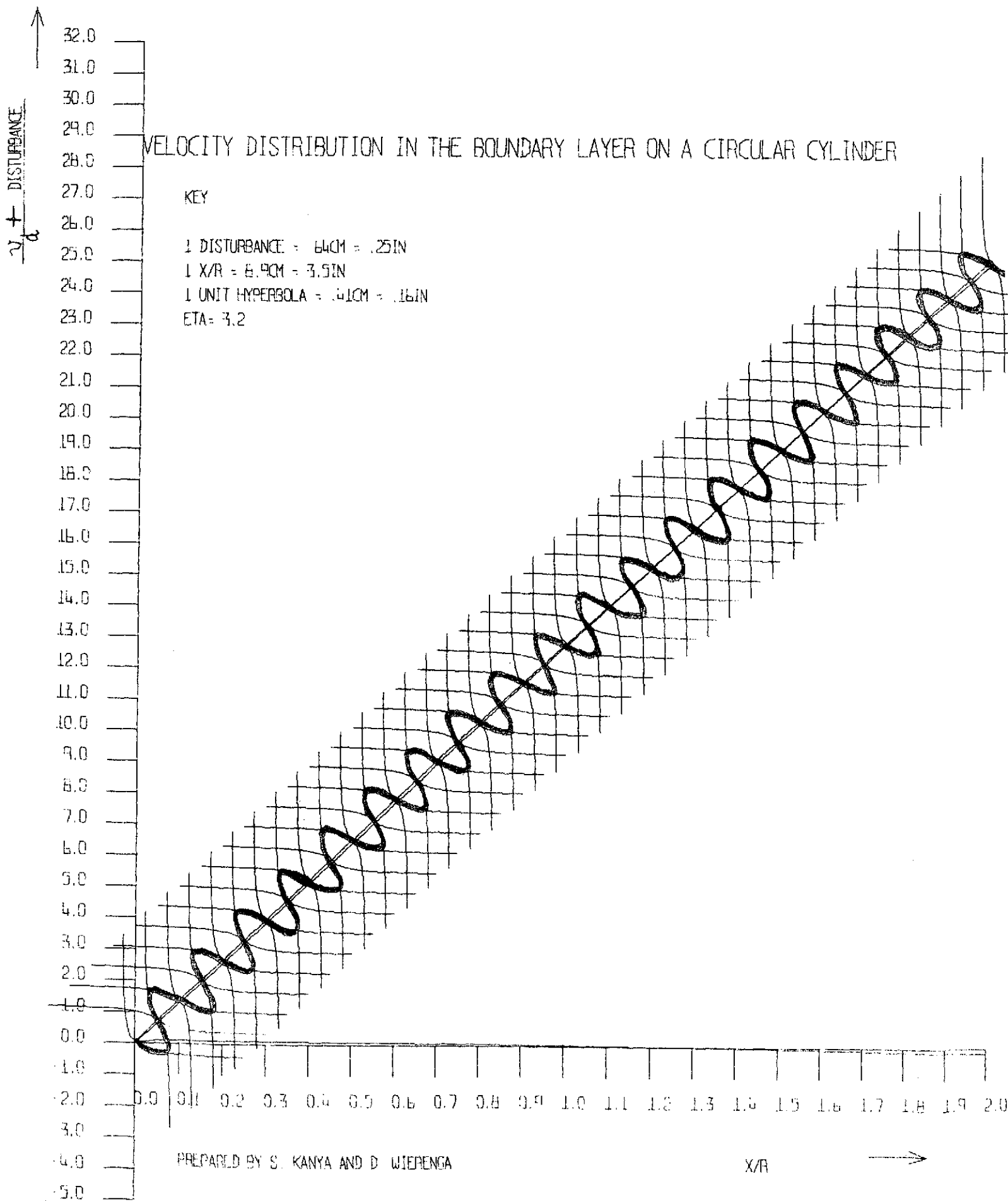
1 DISTURBANCE = .64CM = .25IN

1 X/R = 8.9CM = 3.5IN

1 UNIT HYPERBOLA = .41CM = .16IN

ETA = 3.0





VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ON A CIRCULAR CYLINDER

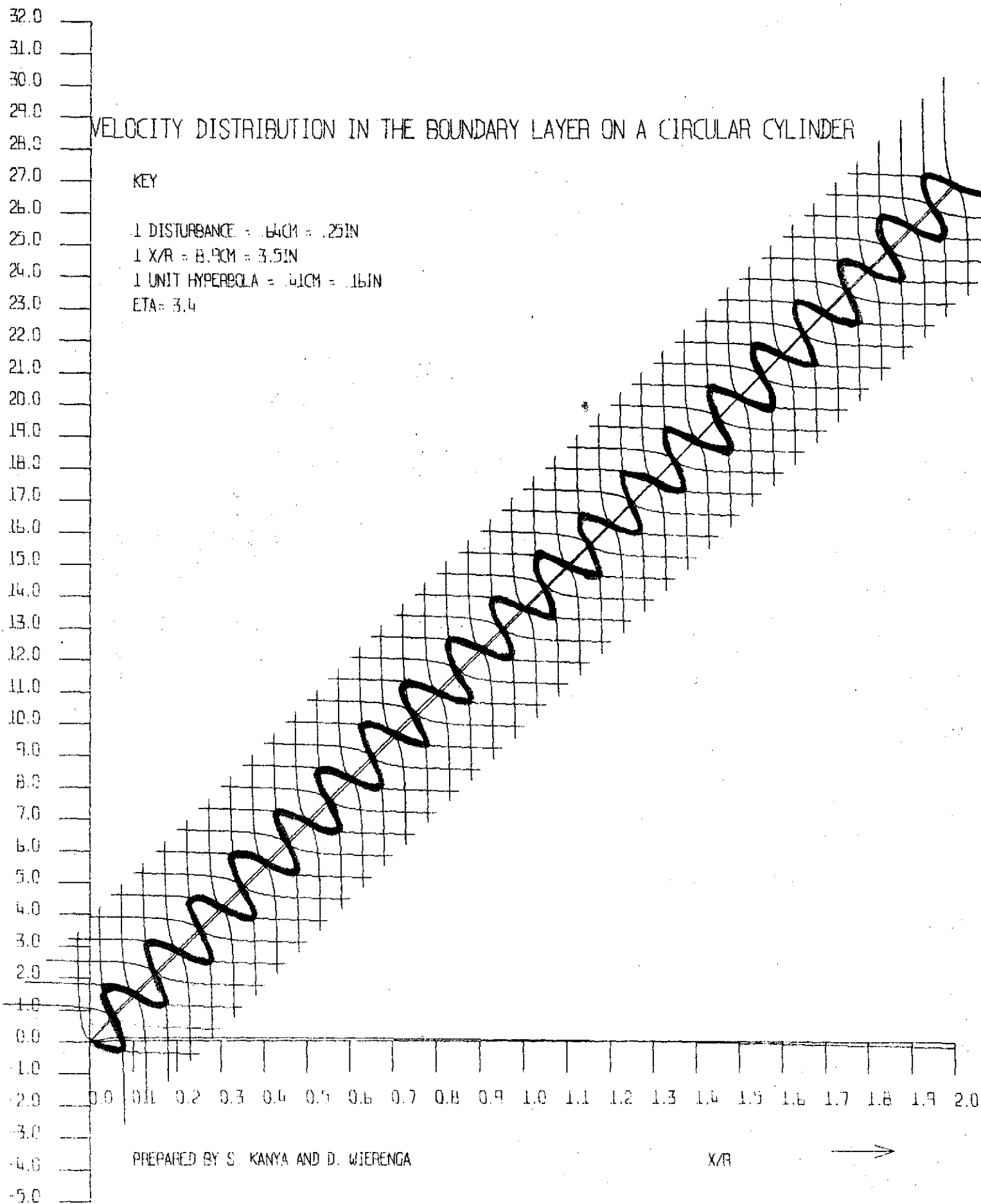
KEY

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1 X/R = $0.9\text{CM} = 3.5\text{IN}$

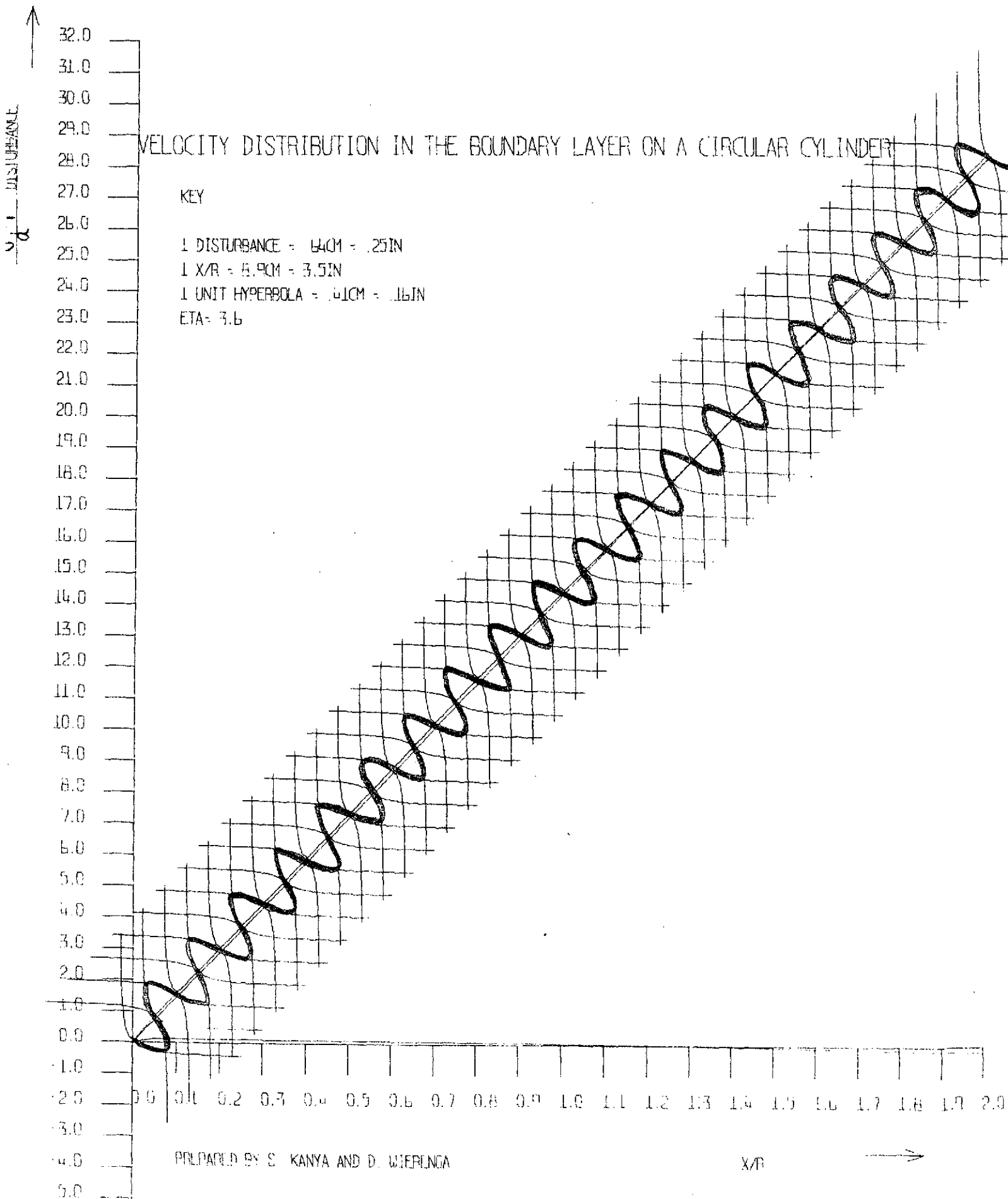
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ETA = 3.4

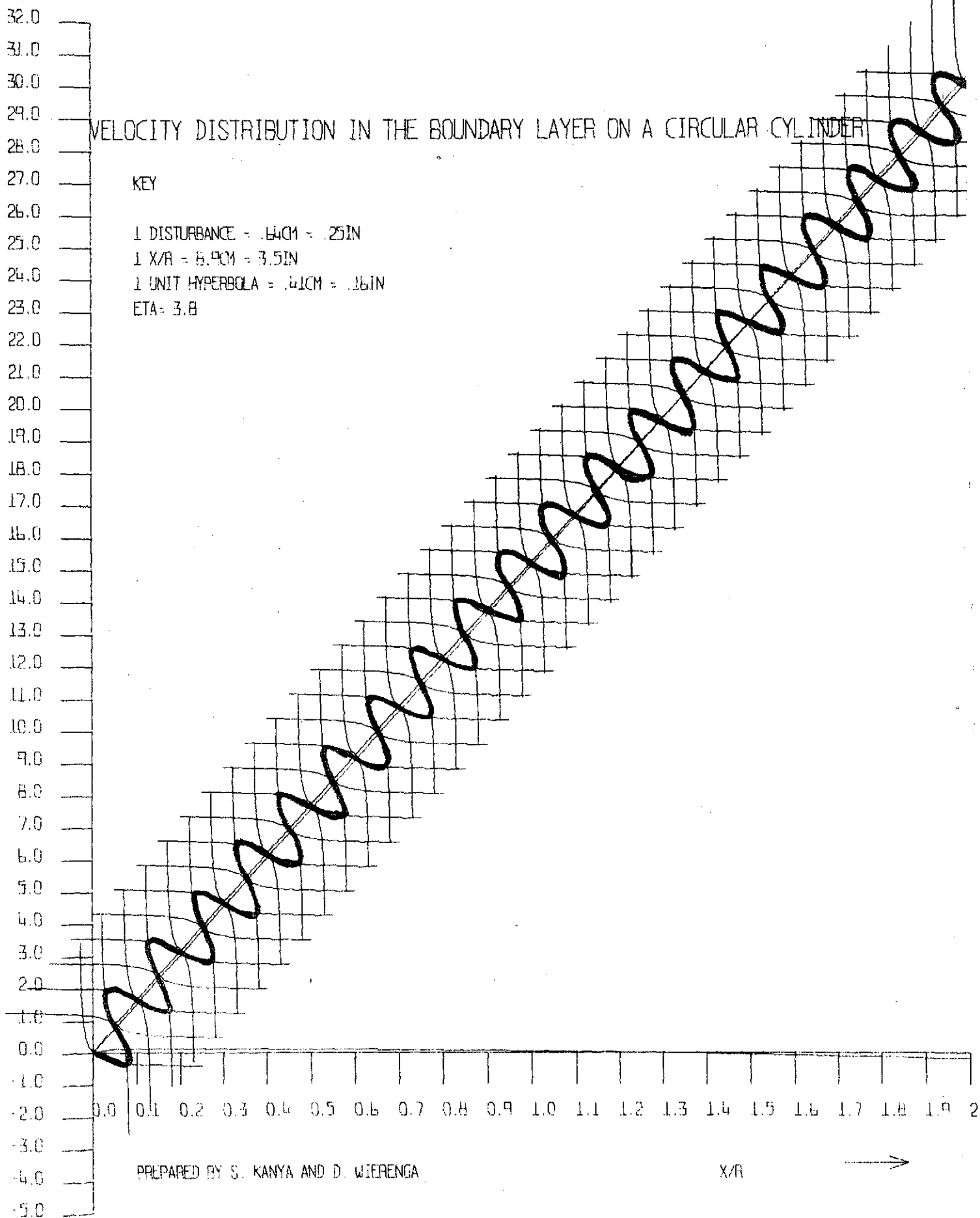


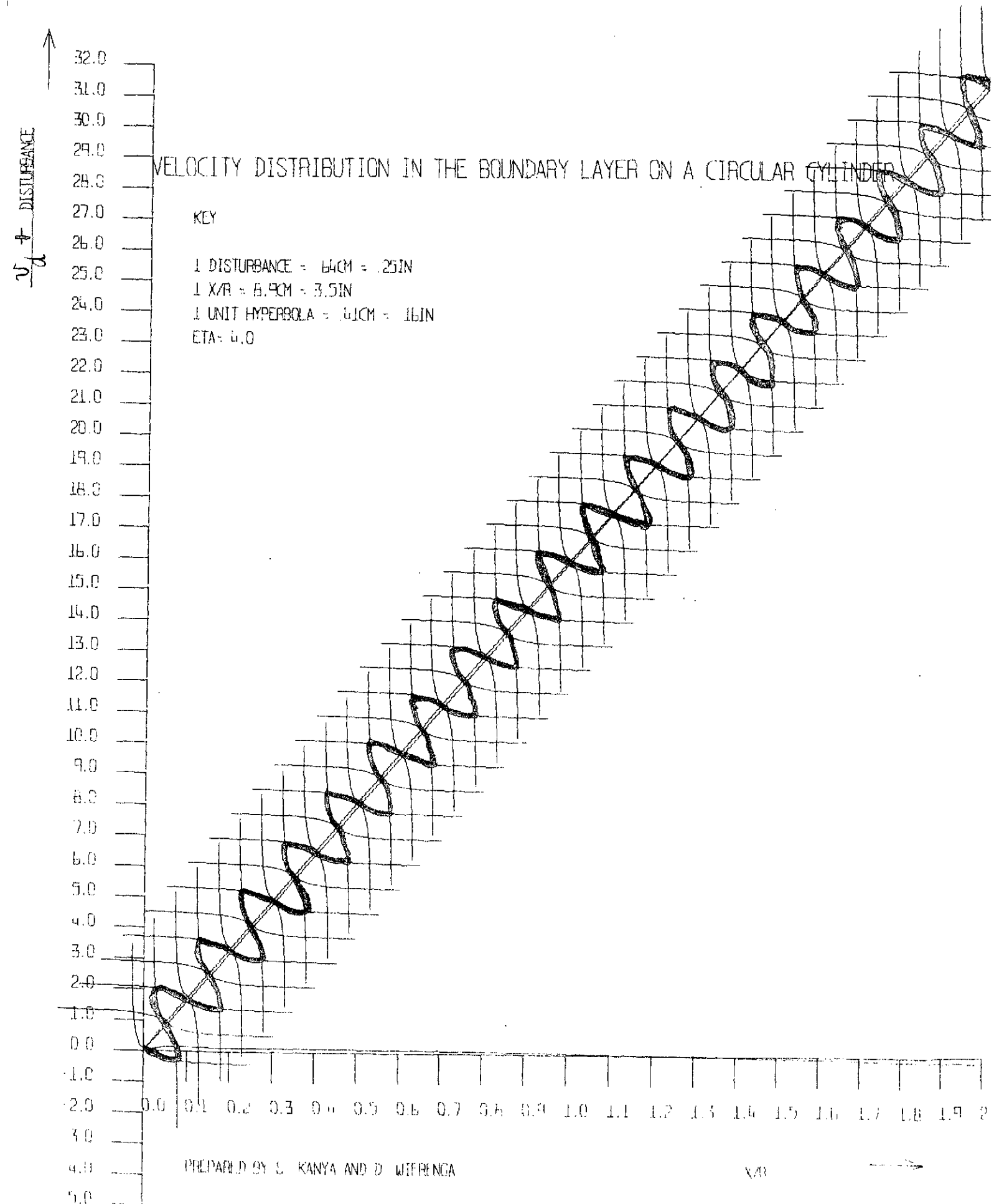
PREPARED BY S. KANYA AND D. WIERENGA

X/R



$v_d + \text{DISTURBANCE}$





APPENDIX

1. OTHER POSSIBLE FORMS

Wentzel (see References) cites the Schroedinger-Gordon wave equation, which has the form:

$$(\square - \mu^2) \psi(x, t) = 0, \quad (1.1)$$

$$\square \equiv \sum_{\nu=1}^4 \partial^2 / \partial x_\nu^2 \equiv \nabla^2 - c^{-2} \partial^2 / \partial t^2, \quad (1.2)$$

where

$$\nu = 1, 2, 3, 4, \quad (1.3)$$

$$x_1, x_2, x_3 = \text{space coordinates}, \quad (1.4)$$

$$x_4 = i c t, \quad (1.5)$$

it is known that the Broglie complex wave functions

$$\exp \{ i (k x + [\mu^2 + k^2 c^2]^{1/2} t) \} \quad (1.6)$$

are particular solutions of the wave equation, (1.1). The scalar constant, $\mu h c^{-1}$, represents the rest mass of the respective particles. It turns out, indeed, that the stationary states of quantized fields obeying the field equations, (1.1), represent systems of particles with the rest mass $\mu h c^{-1}$ (Wentzel, p. 22).

As is known, the association between quantum mechanics and macroscopic hydrodynamics in the classical sense was proposed by Irving Madelung in 1926. Exactly in the same year, Schroedinger announced his famous wave mechanics equation under the name of "amplitude equation." Madelung applies the "amplitude equation" of Schroedinger to the derivation of the equations of motion of the hydrodynamic medium in the classical sense; the conditions and restrictions superimposed upon the Schroedinger equation and next superimposed by Madelung upon the hydrodynamic system in question and the manners of overcoming these difficulties are of the following nature:

(a) The macroscopic hydrodynamic system in question when

moving in one direction should be free from the action of the curl. Of course one can use another system for the curl separately and add the results.

- (b) The Schroedinger equation refers to the amplitude and wave phenomena of one, single electron only. To use it in the sense of the macroscopic fluid dynamics, one has to propose some sort of generalization of the Schroedinger equation to a group or cluster of elements or molecules. Three such possible cases referring to the three possible groups of elements, h, m, were discussed in the present research.

In that respect, the writer repeats the principles of the method from Methods of Quantum Field Theory in Statistical Physics, by A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, Institute for Physical Problems, Academy of Sciences, U.S.S.R., revised English edition, translated and edited by Richard A. Silverman:

"In recent years, remarkable success has been achieved in statistical physics, due to the extensive use of methods borrowed from quantum field theory. The fruitfulness of these methods is associated with a new formulation of perturbation theory, primarily with the application of 'Feynman diagrams.' The basic advantages of the diagram technique lies in its intuitive character: Operating with one-particle concepts, we can use the technique to determine the structure of any approximation, and we can then write down the required expressions with the aid of correspondence rules. These new methods make it possible not only to solve a large number of problems which did not yield to the old formulation of the theory, but also to obtain many new relations of a general character. At present, these are the most powerful and effective methods available in quantum statistics.

There now exists an extensive and very scattered journal literature devoted to the formulation of field theory methods in quantum statistics and their application to specific problems. However, familiarity with these methods is not widespread among scientists working in statistical physics. Therefore, in our opinion, the time has come to present a connected account of this subject, which is both sufficiently complete and accessible to the general reader."

The present writer uses Feynman's (Nobel Prize) exact technique with the correspondence rules properly adjusted to the macroscopic fluid dynamics in two dimensions. The three-dimensional problems are not attacked, as yet, and have to be attacked in the future.

2. HEISENBERG OPERATORS

2.1. Preliminary Remarks

There exists another formalism in attacking the problems in the field of quantum mechanics, i. e., Heisenberg operators. This technique is not used for the time being in the present report and for this reason it will be only briefly mentioned here. May be in the future the writer may propose a method of application of Heisenberg operators to the problem which is under discussion in the present report.

2.2. Remarks on Operators

Suppose that the function $f(x)$ has a power series expansion:

$$f(x) = f_0 + x f_1 + x^2 f_2 + x^3 f_3 + \dots ; \quad (2.2.1)$$

then, we can define the operator $f(A)$ by:

$$f(A) = f_0 + A f_1 + A^2 f_2 + A^3 f_3 + \dots ; \quad (2.2.2)$$

for example, the operator $\exp(\lambda A)$ is:

$$\exp(\lambda A) = 1 + \lambda A + \lambda^2 (2!)^{-1} A^2 + \lambda^3 (3!)^{-1} A^3 + \dots ; \quad (2.2.3)$$

it may be convenient now to introduce the proper notation used in the vector algebra:

a column vector is denoted by: $|\Psi\rangle$;

a row vector is denoted by: $\langle\Psi|$;

examples:

column vector:

$$|\Psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} ; \quad (2.2.4)$$

a row vector:

$$\langle\Psi| = (\psi_x^* \quad \psi_y^*) , \quad (2.2.5)$$

where $*$ stands for complex conjugate; the scalar product of a row vector $\langle\Phi|$ and a column vector $|\Psi\rangle$ is equal to:

$$\langle \Phi | \Psi \rangle = \phi_x^* \psi_x + \phi_y^* \psi_y = \langle \Psi | \Phi \rangle^* ; \quad (2.2.6)$$

if $|\Psi\rangle$ has unit length then:

$$|\psi_x|^2 + |\psi_y|^2 = 1 ; \quad (2.2.7)$$

this is called the normalization condition. As an example consider the following case: all beams of light are superpositions of many beams consisting of one photon each, one may turn his attention to the polarization properties of single photons. It is relatively easy to discover the probability rules for one photon from the knowledge of the behavior of classical beams. The general laws of quantum mechanics are just generalizations of these rules. For one photon one has (Baym, p. 3):

$$|\vec{E}|^2 V = 8 \pi \hbar \omega ; \quad (2.2.8)$$

where:

$\vec{E} = \vec{E}(r, t)$ = electric field vector;

V = volume;

$\hbar = h(2\pi)^{-1}$;

ω = the angular frequency.

One defines the "state vector" of the photon polarization by:

$$|\Psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} , \quad (2.2.9)$$

and by writing:

$$\psi_x = [V(8 \pi \hbar \omega)^{-1}]^{1/2} E_x ; \quad (2.2.10)$$

$$\psi_y = [V(8 \pi \hbar \omega)^{-1}]^{1/2} E_y ; \quad (2.2.11)$$

the $|\Psi\rangle$ vectors are vectors in a "complex" two-dimensional space, their components being complex numbers. From Equation (2.2.7) it follows that $|\Psi\rangle$ has unit length and:

$$|\psi_x|^2 + |\psi_y|^2 = 1 . \quad (2.2.12)$$

In fact the state vectors are independent of the volume V and depend only on the state of polarization of the photon. For example, if:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(i a) \\ \exp(i a) \end{pmatrix} \quad (2.2.13)$$

then the photon is polarized at 45° to the x -axis. A knowledge of the $|\Psi\rangle$ vector gives us all the information we can obtain about the state of the polarization of the photon.

Some special examples of similar vectors are:

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : x\text{-polarization}; \quad (2.2.14)$$

$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : y\text{-polarization}; \quad (2.2.15)$$

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} ; \text{right circular polarization}; \quad (2.2.16)$$

$$|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} ; \text{left circular polarization}. \quad (2.2.17)$$

Let us associate with each column vector $|\Psi\rangle$ a row vector $\langle\Psi|$ which is defined by:

$$\langle\Psi| = (\psi_x^* \quad \psi_y^*), \quad (2.2.18)$$

where $*$ stands for complex conjugate. We may define some operations like the scalar product (discussed already above): scalar product of a row vector $\langle\Phi|$ and a column vector $|\Psi\rangle$:

$$\langle\Phi|\Psi\rangle = \phi_x^* \psi_x + \phi_y^* \psi_y = \langle\Psi|\Phi\rangle^* ; \quad (2.2.19)$$

the known normalization condition requires for the vector $|\Psi\rangle$:

$$\langle\Psi|\Psi\rangle = 1 ; \quad (2.2.20)$$

clearly:

$$\langle x|x\rangle = \langle y|y\rangle = 1 ; \quad (2.2.21)$$

$$\langle R | R \rangle = \langle L | L \rangle = 1 ; \quad (2.2.22)$$

the vectors $|x\rangle$ and $|y\rangle$ are orthogonal, i. e. , perpendicular, in the sense that:

$$\langle x | y \rangle = 0 . \quad (2.2.23)$$

The vectors $|x\rangle$ and $|y\rangle$ form a basis, since any $|\Psi\rangle$ vector can be written as a linear superposition of them:

$$|\Psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \psi_x |x\rangle + \psi_y |y\rangle ; \quad (2.2.24)$$

they are orthogonal, they satisfy the normalization condition, and they form the "orthonormal" basis. Similarly, the set $|R\rangle$ and $|L\rangle$ form an orthonormal basis, since we can always write:

$$|\Psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \frac{\psi_x - i\psi_y}{\sqrt{2}} |R\rangle + \frac{\psi_x + i\psi_y}{\sqrt{2}} |L\rangle . \quad (2.2.25)$$

We can regard any arbitrary polarization as a coherent (logically consistent) superposition of x and y polarization states, or equivalently as a coherent superposition of right and left circularly polarization states. Let us return to the problem of passing a beam through an x-polaroid. The classical rules tell us to regard the beam as a superposition of an x-polarized beam and a y-polarized beam, and that the effect of a polaroid is to throw away the y-polarized component and pass only the x-polarized component. The absolute value squared of the amplitude of the beam gives us its energy before it passes through, and the absolute value squared of its x-component give us its energy after it passes through. The fraction of the beam that passes through is given by equation:

$$|E_x|^2 [|E_x|^2 + |E_y|^2]^{-1} = |E_x|^2 [|\vec{E}|^2]^{-1} ; \quad (2.2.26)$$

as we know, quantum mechanically the fraction (2.2.26) gives us the probability of one photon with the initial polarization passing through the polaroid. Written in terms of $|\psi\rangle$, Eq. (2.2.26) is the "probability":

$$\text{probability} = |\psi_x|^2 [|\psi_x|^2 + |\psi_y|^2]^{-1} = |\psi_x|^2 = |\langle x|\Psi \rangle|^2. \quad (2.2.27)$$

Thus $\langle x|\Psi \rangle$ is the amplitude of the x-polarized component of $|\Psi \rangle$, and its absolute value squared is the probability that the photon in the state $|\Psi \rangle$ passes through the x-polaroid. We call $\langle x|\Psi \rangle$ the "probability amplitude" for the photon to pass through the x-polarizer.

Next, consider passing light through a prism that passes right circular light but rejects left circular light. To calculate the probability, we write the beam as a coherent sum of right circularly polarized and left circularly polarized light, in terms of the state vectors. Then $\langle R|\Psi \rangle$ is the amplitude of the component passed through the prism and $|\langle R|\Psi \rangle|^2$ is the probability that a photon in the state $|\Psi \rangle$ will pass through the prism that passes only the right circular polarized light. Thus the general rule is that if we have a prism that passes only light in the state $|\Phi \rangle$, rejecting light in states orthogonal to $|\Phi \rangle$, then the probability amplitude that a photon in the state $|\Psi \rangle$ will pass through the prism is:

$$\langle \Phi | \Psi \rangle, \quad (2.2.28)$$

and the probability that the photon passes through is:

$$|\langle \Phi | \Psi \rangle|^2. \quad (2.2.29)$$

Notice that the probability is independent of the phase of $|\Psi \rangle$ or $|\Phi \rangle$.

2.3. Equations

Above, we defined the operator $f(A)$ as:

$$f(A) = f_0 + A f_1 + A^2 f_2 + A^3 f_3 + \dots \quad (2.3.1)$$

An application of an operator (2.3.1) to any "state vector," Equation (2.2.9), say, causes a transformation of this vector. The operator, composed of the right and left circular polarizations, is usually denoted by the symbol $R(\theta)$. In general, if we transform a vector $|\Psi \rangle$ by $R(\theta)$, the new components of $|\Psi \rangle$ will bear little resemblance to the old components. Let us ask if there are any state vectors that when operated upon by $R(\theta)$ are at most multiplied by a constant:

$$R(\theta)|\Psi\rangle = C |\Psi\rangle . \quad (2.3.2)$$

If a vector satisfies a relation like this it is called an "eigenvector" of $R(\theta)$, and the number C is called the "eigenvalue" of $R(\theta)$ belonging to the eigenvector in question. Analogously, one can talk about the "eigenstates" of an operator. If $|\alpha\rangle$ is an eigenstate of $f(A)$, Equation (2.3.1), with eigenvalue α , then operating on $|\alpha\rangle$ with $f(A)$, one gets:

$$\begin{aligned} f(A)|\alpha\rangle &= (f_0 + A f_1 + A^2 f_2 + A^3 f_3 + \dots)|\alpha\rangle \\ &= (f_0 + \alpha f_1 + \alpha^2 f_2 + \alpha^3 f_3 + \dots)|\alpha\rangle \\ &= f(\alpha)|\alpha\rangle . \end{aligned} \quad (2.3.3)$$

The above operation emphasizes clearly that $|\alpha\rangle$ is an eigenstate of the operator $f(A)$ with an eigenvalue $f(\alpha)$. The eigenstates of an operator A form the so-called "complete" set of eigenstates of A ; one of the conditions, to be satisfied by the set in question is the following one:

$$1 = \sum_{\alpha} |\alpha\rangle\langle\alpha| ; \quad (2.3.4)$$

the eigenstates of A can always be chosen so that a complete set of A , they form is Hermitian. Operating on both sides of Equation (2.3.4) with $f(A)$ one finds from Equation (2.3.4):

$$f(A) = \sum_{\alpha} f(\alpha) |\alpha\rangle\langle\alpha| . \quad (2.3.5)$$

In fact, one could take this (i. e. , Equation (2.3.5)) as the equation defining $f(A)$ in terms of the eigenstates of A ; it is not necessary for " f " to have a power series expansion to be a usable definition. For illustrative purposes, one can define a delta function of an operator, $\delta(A-x)$, by:

$$\delta(A-x) = \sum_{\alpha} \delta(\alpha-x) |\alpha\rangle\langle\alpha| , \quad (2.3.6)$$

where x is number (Baym, p. 135).

To complete the explanations of the notation, the writer puts down a few more notations used in the text below. Let the letter " l " denote

the length in the z direction, say, on which length some changes in the states of a phenomenon may occur. Let the incident state in a phenomenon in question be denoted by the symbol given below:

$$\text{incident state} = |e\rangle ; \quad (2.3.7)$$

the distance l changes the incident state $|e\rangle$ into the state:

$$|e\rangle \rightarrow \exp(i k_e l) |e\rangle ; \quad (2.3.8)$$

where:

$$k_e = \text{wave number; } (\sim \text{cm}^{-1});$$

example in the electric field:

$$E_x(\vec{r}, t) = E_x^0 \cos(kz - \omega t + \alpha_x) ; \quad (2.3.9)$$

$$E_y(\vec{r}, t) = E_y^0 \cos(kz - \omega t + \alpha_y) ; \quad (2.3.10)$$

$$k = 2\pi \lambda^{-1} = \text{wave number; } (\sim \text{cm}^{-1});$$

$$\omega = \text{angular frequency; } (\sim \text{sec}^{-1});$$

period of a vibratory motion = time required for a complete vibration;

frequency = number of complete vibrations, or cycles, in a unit of time;

the frequency is numerically equal to the reciprocal of the period;

λ = wave length of the harmonic wave = distance between successive points at which the wave differs in phase by 2π , or the distance between maximum displacements or crests; α_x, α_y = the phases; phase is a term stating where the body is at any instant in its vibrations, and what its direction of motion is; the wave numbers with subscripts denote:

$$k_e = \omega n_e c^{-1}; (\sim \text{cm}^{-1}); \quad (2.3.11)$$

$$k_o = \omega n_o c^{-1}; (\sim \text{cm}^{-1}); \quad (2.3.12)$$

where n_e, n_o , are integers. The wave numbers often do not have any special names.

If the phenomenon in question takes place on the length " l ", then the difference in phase of an "e wave" between its point of entry and its point of departure will be " $k_e l$ ". Similarly, an "o wave" changes phase by " $k_o l$ ". The time factor $\exp(-i\omega t)$ is the same for both states (of

polarization, say) and does not effect their relative phase. In Equation (2.3.8) the incident state $|e\rangle$ is changed on the distance ℓ ; this implies that (as mentioned already in Equation (2.3.8)):

$$|e\rangle \rightarrow \exp(i k_e \ell) |e\rangle ; \quad (2.3.13)$$

as it is seen, the function, which introduces (causes) the changes in the states is assumed to be in the form of the exponential function, $\exp(i k_e \ell)$. Similarly, and incident $|0\rangle$ state changes into

$$|0\rangle \rightarrow \exp(i k_0 \ell) |0\rangle ; \quad (2.3.14)$$

the state $|0\rangle$ is associated with the wave number k_0 , i. e.,:

$$\text{state } |0\rangle ; \text{ wave number } k_0 = \omega n_0 c^{-1} ; \quad (2.3.15)$$

let us define the wave number "matrix" by:

$$K = k_e |e\rangle \langle e| + k_0 |0\rangle \langle 0|. \quad (2.3.16)$$

The Hermitian adjoint, K^+ , of the matrix K , Equation (2.3.16), is the matrix:

$$K^+ = \begin{pmatrix} \langle x|K|x\rangle^* & \langle y|K|x\rangle^* \\ \langle x|K|y\rangle^* & \langle y|K|y\rangle^* \end{pmatrix} . \quad (2.3.17)$$

It may happen that in some cases (see Baym, p. 33):

$$K^+ = K ; \quad (2.3.18)$$

K is then said to be Hermitian.

As an application of operator functions, one may cite the following example and result:

Suppose that $|\Psi(0)\rangle$ is the state of a system at time $t = 0$; then at another time t , the state of the system is given by the equation:

$$|\Psi(t)\rangle = \sum_E \exp(-i E t \hbar^{-1}) |E\rangle \langle E|\Psi(0)\rangle , \quad (2.3.19)$$

where the sum is over a complete set of energy eigenstates of the Hamiltonian, H , of the system.

The row vector $\langle \Psi(t) |$ is given in terms of $\langle \Psi(0) |$ by:

$$\begin{aligned} \langle \Psi(t) | &= \langle \Psi(0) | (\exp(i H t \hbar^{-1}))^+ \\ &= \langle \Psi(0) | \exp(-i H t \hbar^{-1}), \end{aligned} \quad (2.3.20)$$

where the hermiticity of H is used (see Baym, p. 136). Using Equation (2.3.5), Equation (2.3.19) is simply:

$$|\Psi(t)\rangle = \exp(-i H t \hbar^{-1}) |\Psi(0)\rangle; \quad (2.3.21)$$

equations (2.3.19) and (2.3.21) are only valid if H does not explicitly depend on time (Baym, p. 135). The expression $|\Psi(t)\rangle$ in this form clearly solves the Schroedinger equation:

$$i \hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = H |\Psi(t)\rangle. \quad (2.3.22)$$

2.4. Heisenberg Representation

One may be often interested in knowing how expectation values of operators, in the state $|\Psi(t)\rangle$, change in course of time. If A is an operator, then its expectation value at time t is:

$$\langle A \rangle_t = \langle \Psi(t) | A | \Psi(t) \rangle. \quad (2.4.1)$$

Assume A not to depend on time explicitly. Using Equation (2.3.20) and (2.3.21) one gets:

$$\langle A \rangle_t = \langle \Psi(0) | \exp(i H t \hbar^{-1}) A \exp(-i H t \hbar^{-1}) | \Psi(0) \rangle; \quad (2.4.2)$$

if we define a time-dependent operator $A(t)$ by:

$$A(t) = \exp(i H t \hbar^{-1}) A \exp(-i H t \hbar^{-1}), \quad (2.4.3)$$

then we can also write:

$$\langle A \rangle_t = \langle \Psi(0) | A(t) | \Psi(0) \rangle. \quad (2.4.4)$$

In this equation one can regard the time development of $\langle A \rangle_t$ as occurring because the operator changes in time, while the state remains the same at all

times. This way of regarding the time development of the system is called the "Heisenberg representation" or "Heisenberg picture." The other way of looking at the time development, as in Equation (2.3.22), where the operators remain constant in time, but the states change according to Equation (2.3.22), is called the "Schroedinger representation" (or picture). At $t = 0$, the states and operators are the same in both representations. Both representations give the same results for time dependent expectation values. We can solve for either the time dependence of the states in the Schroedinger representation, or for the time dependence of the operators in the Heisenberg representation.

In the present report the writer applies the Schroedinger representation to the problem proposed and outlined by Madelung, i.e., the association between the quantum mechanics and macroscopic hydrodynamics.